Relative categoricity for finitely generated fields

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Question of categoricity

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Our answer will depend on the meaning of logical theory and of determine.

Second-order categoricity

- $(\mathbb{N}, +, \times, \leq, 0, 1)$ is the unique (up to isomorphism) ordered semi-ring satisfying full induction.
- $(\mathbb{R}, +, \times, \leq, 0, 1)$ is the unique complete Archimedean ordered field.

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Absolute first-order categoricity

It follows from the Löwenheim-Skolem theorems, that if a structure is categorical in first-order logic, then it is finite.

λ-categoricity

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Definition

For λ a cardinal and $\mathfrak{M}$ an $\mathscr{L}$-structure of cardinality λ, we say that $\mathfrak{M}$ is λ-categorical or is categorical in power if for any other $\mathscr{L}$-structure $\mathfrak{N}$ of cardinality λ we have $\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathfrak{M} \cong \mathfrak{N}$.

Limits of categoricity in power

- In a countable language, if $\mathfrak{M}$ is $\aleph_0$-categorical, then there are only finitely many definable sets in M^n for each natural number n.
- Consequently, there are no $\aleph_0$-categorical fields.
- In general, no order is interpretable in an uncountably categorical structure.
- (Uncountable) algebraically closed fields are categorical in power, and are, in fact, the only infinite fields which are categorical in power.

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Relative first-order categoricity

Definition

Fix a first-order language $\mathscr{L}$ and $\mathscr{C}$ a class of $\mathscr{L}$-structures. We say that $\mathfrak{M} \in \mathscr{C}$ is categorical relative to $\mathscr{C}$ if for any $\mathfrak{N} \in \mathscr{C}$ we have $\mathfrak{N} \equiv \mathfrak{M} \Leftrightarrow \mathfrak{N} \cong \mathfrak{M}$.

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λ-categoricity is an instance of relative categoricity if one takes $\mathscr{C} = \mathscr{C}_\lambda := \{\mathfrak{M} \; : \; ||\mathfrak{M}|| = \lambda, \mathfrak{M}$ an $\mathscr{L}$-structure $\}$.

Examples classes studied

- In the more abstract study of non-first-order classification theory, one might restrict $\mathscr{C}$ to be a pseudo-elementary class or even the elements of some pseudo-elementary class omitting a given set of types.
- In common mathematical practice, one restricts the class of structures studied to some class of "standard" objects.
- For example, when studying groups one might study only finitely presented groups, with topological spaces, only smooth manifolds admiting a finite covering by standard coördinate neighborhoods.

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Pop's conjecture

Conjecture (Pop)

Elementarily equivalent finitely generated fields are isomorphic.

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Elementarily equivalent finitely generated fields are isomorphic.

Reformulated in terms of relative categoricity, a finitely generated fields is relatively catefogorical within the class of finitely generated fields.

Related questions

- Sabbagh asked in the mid 80s whether a finitely generated field of transcendence degree one over the rationals could have the same theory as one of transcendence degree two. The resolution of this question is a key step in the solution of Pop's conjecture.
- Oger has shown how to deduce a positive answer to the corresponding question for finitely generated commutative rings from Pop's conjecture.
- Nies, Khelif, Oger, and others have studied the extent to which the isomorphism type of a finitely generated group is determined by its first-order theory. Here, the conjectural solution is more complicated.

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Definability, rather than categoricity

While Pop's conjecture is stated in terms of isomorphism types and in our reformulation in terms of relative categoricity, it is better understood as a question about first-order definability within the class of finitely generated fields.

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While Pop's conjecture is stated in terms of isomorphism types and in our reformulation in terms of relative categoricity, it is better understood as a question about first-order definability within the class of finitely generated fields.

For instance, one could answer Sabbagh's question positively by showing that, relative to the class of finitely generated fields, those finitely generated fields of transcendence degree one form an elementary class.

From $\mathscr{L}_{\omega_1,\omega}$ to $\mathscr{L}_{\omega,\omega}$

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For instance, the field K has transcendence degree zero just in case

$$K \models (\forall y) \bigvee_{Q(X) \in \mathbb{Z}[X]} (Q(y) = 0 \ \wedge \ (\exists z) Q(z) \neq 0)$$

Defining finite

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Proof.

Let $\phi := (\exists u)(\forall x)(\exists y)[x = uy^2 \ \vee \ x = y^2]$. $\square$

Recognizing transcendence degree

Theorem (Poonen, after Pop)

For each natural number n there is a formula $\psi_n(x_1, \ldots, x_n)$ in the language of rings for which if K is a finitely generated field and $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ is an n-tuple from K, then $K \models \psi_n(\mathbf{a})$ if and only if $(a_1, \ldots, a_n)$ is algebraically dependent over the prime field.

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As this is a very strong theorem, one might expect that it requires some deep results from algebra.

Constructing the formula ψ_n

- We restrict to characteristic zero.
- For $\mathbf{b} = (b_1, \ldots, b_d)$ some d-tuple from a field, let $q_{\mathbf{b}} := \sum_{i_1=0}^{1} \cdots \sum_{i_d=0}^{1} b_1^{i_1} \cdots b_d^{i_d} X_{\mathbf{i}}^2$.
- Using Voevodsky's Theorem (Milnor's Conjecture) relating Milnor's K-theory to Galois cohomology, Pop shows that $\mathbf{b} \in K^d$ is a transcendence basis if and only if there are α and $\beta \in K$ algebraic over the rationals for which the equation $q_{(\mathbf{b},\alpha,\beta)}(X) = \gamma$ always has a solution in $K[\sqrt{-1}]$ while the only solution to $q_{(\mathbf{b},\alpha,\beta)}(X) = 0$ is the zero vector.
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Using ψ_n

- If K is finitely generated, then the constant field, k, of K is defined by $k = \psi_1(K) := \{a \in K \mid K \models \psi_1(a)\}$.
- The finitely generated field K has positive characteristic if and only if the sentence ϕ relativized to $\psi_1(K)$ holds.
- If K is a finitely generated field and $\mathbf{t} = (t_1, \ldots, t_n) \in K^n$ is algebraically independent, then the relative algebraic closure of the subfield generated by $\mathbf{t}$ is defined by $\psi_{n+1}(x, \mathbf{t})$.

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J. Robinson's definition of $\mathbb{Z}$

Theorem (J. Robinson)

There is a formula $\zeta(x)$ in the language of rings for which $\mathbb{Z} = \zeta(\mathbb{Q}) := \{a \in \mathbb{Q} \mid \mathbb{Q} \models \zeta(a)\}$.

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Theorem (J. Robinson)

If K is a number field, then $\mathscr{O}_K$, the ring of algebraic integers of K, is definable in K and $\mathbb{Z}$ is definable in $\mathscr{O}_K$.

R. Robinson's interpretation of $\mathbb{Z}$

Theorem (R. Robinson)

Let k be a finite field and K a finitely generated extension field of k having transcendence degree one. There is a formula $\mu(x, y, z, w)$ so that for any transcendental element $t \in K \smallsetminus k$, the set of triples $\{(t^n, t^m, t^{nm}) \mid n, m \in \mathbb{Z}\}$ is defined by $\mu(x, y, z; t)$.

Rumely's biïnterpretation

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Using this theorem of Rumely and Poonen's ψ_1 and ψ_2, we find that $(\mathbb{Z}, +, \times)$ is uniformly interpretable in the class of infinite finitely generated fields.

Interpreting fields in $\mathbb{Z}$

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Theorem (Gödel)

There are definable functions $\oplus : \mathbb{Z}^3 \to \mathbb{Z}$ and $\otimes : \mathbb{Z}^3 \to \mathbb{Z}$ for which for any integer $a \in \mathbb{Z}$ the structure $K_a := (\mathbb{Z}, \oplus_a, \otimes_a)$ is a finitely generated field and if K is any infinite finitely generated field, then there is some integer $[K] \in \mathbb{Z}$ for which $(K, +, \times) \cong K_{[K]}$.

Biïnterpretation?

If K is an infinite finitely generated field, then $\mathbb{Z}$ is (possibly, parametrically) interpretable in K and K is interpretable in $\mathbb{Z}$. Do these interpretations form a biïnterpretation?

- The answer to the first of the questions is easily seen to be yes as the isomorphism in question is recursive.
- Since $\mathbb{Z}$ is rigid while K may have automorphisms, it is essential to allow parameters in any definition of an isomorphism between K and $K_{[K]}$. With this proviso, the answer to the second question is also yes, but the proof is more difficult.

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Defining the isomorphism

- In the case of global fields, Rumely already observed that his uniform definition of the valuations yields a uniform internal Gödel coding.
- In the case of higher transcendence degree, one achieves the biïnterpretation by giving an internal definition of evaluation of elements of the field considered as a function field.

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Theory from biïnterpretation

Proposition

Let $\mathscr{C}$ be a class of recursively presented structures in a finite language. Suppose that $(\mathbb{Z}, +, \times)$ is uniformly interpreted in $\mathscr{C}$ and that the structure $\mathfrak{M} \in \mathscr{C}$ is parametrically biïnterpretable with $\mathbb{Z}$ via the above uniform interpretation. Then there is a single sentence $\xi_{\mathfrak{M}}$ for which $\mathfrak{M}$ is the only element of $\mathscr{C}$ satisfying ξ_M up to isomorphism.

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Applying this proposition to the class of infinite finitely generated fields, we see that the isomorphism type of any finitely generated field is determined by a single sentence.

QFA

In the work of Nies, et al, on the theories of finitely generated groups, the property of a structure having its theory isolated by a single sentence relative to some class of structures is called quasi-finite axiomatizability (QFA).

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As this term is already used in connection with Zilber's work on totally categorical theories, an alternative should be used and I suggest relatively finitely axiomatizable would be a better word.

Uniform definability

If θ is a sentence in the language of rings, then the set $[\theta] := \{a \in \mathbb{Z} \mid K_a = (\mathbb{Z}, \oplus_a, \otimes_a) \models \theta\}$ is an arithmetic set.

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Question

If $X \subseteq \mathbb{Z}$ is an arithmetic set which is closed under isomorphism in the sense that $(a \in X \text{ and } K_a \cong K_b) \Rightarrow b \in X$, then is there some sentence θ for which $X = [\theta]$?

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It follows from the biïnterpretability that there is some countable (even arithmetic) set of sentences Θ for which $X = \bigcap_{\theta \in \Theta} [\theta]$.

Geometric problem

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Question

The structures $(\mathbb{Z}, +, \times)$ and $(\mathbb{Q}^{\mathrm{alg}}(s, t), +, \times)$ interpret each other. Are they biïnterpretable?