Proof mining in $\mathbb{R}$-trees and hyperbolic spaces

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General metatheorem

Theorem 1 (Gerhardy/Kohlenbach, 2005)

P Polish space, K compact metric space, ρ "small" type, $B_\forall(x^\rho, n^0)$, $C_\exists(x^\rho, m^0)$ contain only x, n free, resp. x, m free. Assume that

$$\mathcal{A}^\omega[X,d]_{-b} \vdash \forall z \in P \forall y \in K \forall x^\rho (\forall n B_\forall(x,n) \to \exists m C_\exists(x,m)).$$

Then there exists a computable functional $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N}^{(\mathbb{N}\times\ldots\mathbb{N})} \to \mathbb{N}$ such that the following holds in all nonempty metric spaces (X,d):

for all representatives r_z of $z \in P$ and all $x \in S_\rho, x^* \in \mathbb{N}^{(\mathbb{N}\times\ldots\mathbb{N})}$, if there exists an $a \in X$ such that $x^* \gtrsim_\rho^a x$, then

$$\forall y \in K \big(\forall n \leq \Phi(r_z, x^*) B_\forall(x,n) \to \exists m \leq \Phi(r_z, x^*) C_\exists(x,m)\big).$$

The theorem also holds for nonempty hyperbolic spaces (X,d,W), CAT(0)-spaces, normed spaces, inner product spaces.

General metatheorem

- the metatheorem can be used as a black box: infer new uniform existence results without any proof analysis
- run the extraction algorithm:
    - extract an explicit effective bound
    - given proof $p \Rightarrow$ new proof p^* for the stronger result
    - new mathematical proof of a stronger statement which no longer relies at any logical tool

Metatheorems for other classes of spaces

- adapt the metatheorem to other classes of spaces:

1. the language may be extended by a-majorizable constants

2. the theory may be extended by purely universal axioms

Gromov hyperbolic spaces

(X, d) metric space

• the *Gromov product* of x and y with respect to the *base point* w is defined to be:

$$(x \cdot y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).$$

• Let $\delta \geq 0$. X is called $\delta - hyperbolic$ if for all $x, y, z, w \in X$,

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta.$$

X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

• X is $\delta - hyperbolic$ iff for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta.$$

Gromov hyperbolic spaces

• The theory of Gromov hyperbolic spaces, $\mathcal{A}^\omega[X, d, \delta\text{-hyperbolic}]_{-b}$ is defined by extending $\mathcal{A}^\omega[X, d]_{-b}$ as follows:

1. add a constant δ^1 of type 1,
2. add the axioms

   $\delta \geq_{\mathbb{R}} 0_{\mathbb{R}}$,

   $\forall x^X, y^X, z^X, w^X \big(d_X(x,y) +_{\mathbb{R}} d_X(z,w) \leq_{\mathbb{R}}$

   $\leq_{\mathbb{R}} \max_{\mathbb{R}}\{d_X(x,z) +_{\mathbb{R}} d_X(y,w), d_X(x,w) +_{\mathbb{R}} d_X(y,z)\} +_{\mathbb{R}} 2 \cdot_{\mathbb{R}} \delta\big)$

• Theorem 1 holds also for $\mathcal{A}^\omega[X, d, \delta\text{-hyperbolic}]_{-b}$ and nonempty Gromov hyperbolic spaces (X, d, δ)

W-hyperbolic spaces

[Takahashi, Goebel/Kirk, Reich/Shafrir, Kohlenbach]

A W**-hyperbolic space** is a triple (X,ρ,W) where (X,d) is metric space and $W: X\times X\times[0,1]\to X$ s.t.

$$\begin{array}{ll}(W1) & d(z,W(x,y,\lambda))\le(1-\lambda)d(z,x)+\lambda d(z,y),\\ (W2) & d(W(x,y,\lambda),W(x,y,\tilde\lambda))=|\lambda-\tilde\lambda|\cdot d(x,y),\\ (W3) & W(x,y,\lambda)=W(y,x,1-\lambda),\\ (W4) & d(W(x,z,\lambda),W(y,w,\lambda))\le(1-\lambda)d(x,y)+\lambda d(z,w).\end{array}$$

Notation: $(1-\lambda)x\oplus\lambda y := W(x,y,\lambda)$.
$[x,y] := \{W(x,y,\lambda) : \lambda\in[0,1]\}$.

$\mathbb{R}$-trees

• $\mathbb{R}$-trees introduced by Tits('77)

• A *geodesic* in a metric space (X, d) is a map $\gamma : [a, b] \to \mathbb{R}$ s.t. for all $s, t \in [a, b]$,

$$d(\gamma(s), \gamma(t)) = |s - t|$$

X is said to be a *geodesic space* if every two points are joined by a geodesic.

• A metric space (X, d) is an $\mathbb{R}$-*tree* if X is a geodesic space containing no homeomorphic image of a circle.

• X is an $\mathbb{R}$-tree $\Leftrightarrow$ X is a 0-hyperbolic geodesic space $\Leftrightarrow$ X is a W-hyperbolic space satisfying

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\}.$$

$\mathbb{R}$-trees

- $\mathcal{A}^\omega[X, d, W, \mathbb{R}\text{-tree}]_{-b}$ results from $\mathcal{A}^\omega[X, d, W]_{-b}$ by adding the axiom:

$$\begin{cases} \forall x^X, y^X, z^X, w^X \big(d_X(x,y) +_{\mathbb{R}} d_X(z,w) \leq_{\mathbb{R}} \\ \qquad\qquad \leq_{\mathbb{R}} \max_{\mathbb{R}}\{d_X(x,z) +_{\mathbb{R}} d_X(y,w), d_X(x,w) +_{\mathbb{R}} d_X(y,z)\}\big). \end{cases}$$

- Theorem 1 holds also for $\mathcal{A}^\omega[X, d, W, \mathbb{R}\text{-tree}]_{-b}$ and nonempty $\mathbb{R}$-trees.

Uniformly convex W-hyperbolic spaces

(X, ρ, W) is *uniformly convex* if for any $r > 0$, and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ s. t. for all $a, x, y \in X$,

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r. \qquad (1)$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called a *modulus of uniform convexity*.

Uniformly convex W-hyperbolic spaces

• The theory $\mathcal{A}^\omega[X, d, W, \eta]_{-b}$ of uniformly convex W-hyperbolic spaces extends the theory $\mathcal{A}^\omega[X, d, W]_{-b}$ as follows:

1. add a new constant η_X of type 000,
2. add the following axioms:

$$\begin{cases} \forall r^0 \forall k^0 \forall x^X, y^X, a^X \big(d_X(x, a) <_{\mathbb{R}} r \ \wedge \ d_X(y, a) <_{\mathbb{R}} r \wedge \\ \wedge\, d_X(W_X(x, y, 1/2), a) >_{\mathbb{R}} 1 - 2^{-\eta_X(r,k)} \to d_X(x, y) \leq_{\mathbb{R}} 2^{-k} \big), \end{cases}$$

$$\forall r^0, k^0 (\eta_X(r, k) =_0 \eta_X(q(r), k)).$$

• Theorem 1 holds also for $\mathcal{A}^\omega[X, d, W, \eta]_{-b}$ and nonempty uniformly convex W-hyperbolic spaces (X, d, W, η)

Fixed point theory of nonexpansive mappings

(X, d, W) W-hyperbolic, $C \subseteq X$ convex, $(\lambda_n)_{n\in\mathbb{N}}$ sequence in $[0,1]$

• $T : C \to C$ *nonexpansive* if for all $x, y \in C$

$$d(Tx, Ty) \leq d(x, y),$$

• The *Krasnoselski-Mann iteration* starting from $x \in C$:

$$x_0 := x, \;\; x_{n+1} := (1-\lambda_n)x_n \oplus \lambda_n T x_n$$

• asymptotic regularity - defined by Browder/Petryshyn(66) for normed spaces:

T is λ_n-*asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Fixed point theory for nonexpansive mappings

Theorem Browder-Göhde-Kirk

$(X, \|\cdot\|)$ uniformly convex Banach space, $C \subseteq X$ non-empty convex, closed and bounded, $T : C \to C$ nonexpansive.
Then T has a fixed point.

Theorem Ishikawa '76

$(X, \|\cdot\|)$ Banach space, $C \subseteq X$ a nonempty convex bounded subset, $T : C \to C$ nonexpansive.
Suppose that (λ_n) is divergent in sum and $\limsup_{n\to\infty} \lambda_n < 1$.
Then T is λ_n-asymptotically regular.

Groetsch's Theorem

Theorem

(X, d, W, η) uniformly convex W-hyperbolic space s.t η decreases with r (for a fixed ε), $C \subseteq X$ nonempty convex,
$T : C \to C$ nonexpansive s.t. $Fix(T) \neq \emptyset$,
$(\lambda_n) \subseteq [0, 1]$ satisfying

$$\sum_{k=0}^{\infty} \lambda_k(1 - \lambda_k) = \infty.$$

Then T is λ_n-asymptotically regular.

• version for uniformly convex W-hyperbolic spaces of an important theorem for Banach spaces proved by Groetsch(72)

Logical analysis

$\mathcal{A}^\omega[X,d,W,\eta]_{-b}$ proves:

$$\forall(\lambda_n) \subseteq [0,1]\,\forall x \in X, T: X \to X \,\Big(Mon(\eta,r) \wedge T\,n.e. \wedge Fix(T) \neq \emptyset \\ \wedge \sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = \infty \to \lim d(x_n, Tx_n) = 0\Big)$$

$\Updownarrow$

$$\forall(\lambda_n) \subseteq [0,1]\,\forall x \in X, T: X \to X \,\Big(Mon(\eta,r) \wedge T\,n.e. \wedge Fix(T) \neq \emptyset \\ \wedge \exists \theta : \mathbb{N} \to \mathbb{N} \forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i)) \to \lim d(x_n, Tx_n) = 0\Big)$$

Logical analysis

$\Updownarrow$

$$\forall \theta : \mathbb{N} \to \mathbb{N} \forall \lambda_{(\cdot)} \in [0,1]^{\mathbb{N}} \, \forall x \in X, T : X \to X$$
$$\big(T\, n.e. \wedge Fix(T) \neq \emptyset \wedge \ Mon(\eta, r) \ \wedge \forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i))$$
$$\to \forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall \, n \geq N(d(x_n, Tx_n) < 2^{-k})$$

$\Updownarrow$

$$\forall k \in \mathbb{N} \forall \theta : \mathbb{N} \to \mathbb{N} \forall \lambda_{(\cdot)} \in [0,1]^{\mathbb{N}} \, \forall x \in X, T : X \to X$$
$$\big(T\, n.e. \wedge Fix(T) \neq \emptyset \wedge \ Mon(\eta, r) \ \wedge \forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i))$$
$$\to \exists N \in \mathbb{N}(d(x_N, Tx_N) < 2^{-k})\big)$$

Logical analysis

$$\forall k^0\, \forall \theta^1\, \forall \lambda_{(\cdot)}^{0\to 1}\, \forall x^X, T^{X\to X}\big(T\, n.e.\ \wedge\ Fix(T) \neq \emptyset\ \wedge Mon(\eta, r) \wedge$$

$$\wedge\, \forall n^0 (n \leq_{\mathbb{R}} \sum_{i=0}^{\theta(n)} \lambda_i (1-\lambda_i)) \to \exists N^0 (d_X(x_N, T(x_N)) <_{\mathbb{R}} 2^{-k})\big),$$

where $\lambda_{(\cdot)}^{0\to 1}$ represents an element of the compact metric space $[0,1]^{\mathbb{N}}$ with the product metric.

Concrete consequence of metatheorem

Corollary

P Polish space, K compact Polish space, $B_\forall$, and $C_\exists$ be as in Theorem 1. Assume that $\mathcal{A}^\omega[X,d,W,\eta]$ proves that

$$\forall z \in P \forall y \in K \forall x^X, T^{X\to X} \big(T\, n.e. \wedge Fix(T) \neq \emptyset \wedge \forall n^0 B_\forall \to \exists N^0 C_\exists\big),$$

then there exists a computable functional $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{N}$ s.t.

$$\forall r_z \in \mathbb{N}^\mathbb{N} \forall b \in \mathbb{N} \forall y \in K \forall x^X, T^{X\to X} \big(T\, n.e. \wedge$$
$$\wedge \forall \delta > 0 (Fix_\delta(T,x,b) \neq \emptyset) \wedge \forall n^0 B_\forall \to \exists N \leq_0 \Phi(r_z, b, \eta_X) C_\exists\big)$$

holds in any nonempty uniformly convex W-hyperbolic space.

$$Fix_\delta(T,x,b) := \{y^X \mid d_X(y,T(y)) \leq_\mathbb{R} \delta \wedge d_X(x,y) \leq_\mathbb{R} b\}.$$

Logical analysis

Corollary yields the existence of a computable functional $\Phi(k,\theta,b,\eta)$ such that for all $(\lambda_n)\in[0,1]^{\mathbb{N}}, x\in X, T:X\to X$,

$$T\; n.e. \wedge Mon(\eta,r)\wedge \forall\delta>0(Fix_\delta(T,x,b)\neq\emptyset)\wedge$$
$$\wedge\forall n(n\leq\sum_{i=0}^{\theta(n)}\lambda_i(1-\lambda_i))\to\exists N\leq\Phi(k,\theta,b,\eta)(d(x_N,T(x_N))\leq 2^{-k})$$

holds in any nonempty uniformly convex W-hyperbolic space (X,d,W,η).

Bounds on asymptotic regularity

Theorem

(X, d, W, η) uniformly convex W-hyperbolic space s.t η decreases with r, $C \subseteq X$ convex bounded subset with diameter d_C, $T : C \to C$ n. e. $(\lambda_n) \subseteq [0,1]$, $\theta : \mathbb{N} \to \mathbb{N}$ s. t. $\forall n \in \mathbb{N} \left(\sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i) \geq n \right)$.

Then T is λ_n-asymptotic regular and moreover

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, \theta, d_C, \eta) \left(\rho(x_n, Tx_n) \leq \varepsilon \right).$$

• for uniformly convex normed spaces: Kohlenbach, J. Math. Anal. and Appl.(03)

Bounds on asymptotic regularity

L., J. Math. Anal. and Appl. (to appear).

• extraction of $\Phi(\varepsilon,\theta,d_C,\eta)$:

$$\Phi(\varepsilon,\theta,d_C,\eta) := \begin{cases} \theta\left(\left\lceil \dfrac{d_C+1}{\varepsilon\cdot\eta\left(d_C+1,\dfrac{\varepsilon}{d_C+1}\right)}\right\rceil\right) & \text{for } \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

• quadratic rate of asymptotic regularity for CAT(0)-spaces and $\mathbb{R}$-trees

$$\Phi(\varepsilon,d_C,\lambda) := \begin{cases} \dfrac{1}{\lambda(1-\lambda)}\left\lceil \dfrac{4(d_C+1)^2}{\varepsilon^2}\right\rceil & \text{for } \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

Effective bounds for asymptotic regularity

	$\lambda_n = \lambda$	general λ_n
Hilbert	quadratic: Browder/Petryshyn(67)	$\theta\left(\frac{1}{\varepsilon^2}\right)$: K.(03)
UC normed	K.(03), Kirk/Martinez(90)	K.(03)
normed	quadratic: Baillon/Bruck(96)	K.(01)
$\mathbb{R}$-trees, CAT(0)	**quadratic**: L.	$\theta\left(\frac{\mathbf{1}}{\varepsilon^{\mathbf{2}}}\right)$: L.
UC W-hyperbolic	L.	L.
W-hyperbolic	exponential: K./L.(03)	K./L.(03)