Interval Model to Mathematical Morphology for dealing with Uncertainty between two Values of Gray-scale Images

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Abstract The area of digital images processing is an area that is standing out since these last years. There are a number of approaches and methodologies, more and more refined and sophisticated on the treatment and analysis of images. Among them, one can cite those areas realized by means of morphological mathematics, neural nets or fuzzy logic. Sometimes, when someone makes a study on the assessment of images to specific diagnosis, as, for instance, a medical image or a region on a map, a crying need of accuracy is notable, because certainty is a fundamental factor in these cases. In this paper we propose an algebraic model for two valued (l and L) gray-scale images with uncertainty pixels; which, sometimes, we call ***undefined images***. This approach is part of an extension to gray scale images, where the uncertainty on some coordinate will be coded by an interval $[l_1, l_2]$. This algebraic model will to deal with uncertainties sometimes present in images due to different factors. We show the consequences of such introduction in terms of the basic algebraic operations as well as it's influence on morphological operations.

key words: uncertainty, interval, morphology, lattices .

Introduction

In this work we propose an algebraic model for two valued (l and L) gray-scale images with undefined information (uncertainty) for some pixels. The approach here provides the basis for an extension to gray scale images, where an uncertainty will be coded by an interval $[l_1, l_2]$. For example, if at some coordinate x there is an uncertainty if the pixel value is 50 or 51, then it will be coded as the mapping $x \mapsto [50, 51]$. The coding of such uncertainty by just considering a new gray-scale λ to represent the uncertainty, does not solve the problem, since λ would be used to codify different situations; for example, $x_1 \mapsto [50, 51]$ and $x_2 \mapsto [75, 76]$ would be coded as $x_1 \mapsto \lambda$ and $x_2 \mapsto \lambda$, leading to a loss of information. Such ideas belong to the notion of interval image and can be seen at [1, 7]. The type of images we consider in this work is of the form $f : E \to \{[l, l], [l, L], [L, L]\}$, which can be seen as ternary images. We could treat the set of pixel values $\{[l, l], [l, L], [L, L]\}$ as three gray-scales as in standard gray-scale morphology, but this would imply at least a change of spaces (between intervals and gray-scales). Our purpose is then to define the basic operators such as dilations and erosions directly for the interval setting. These images constitute a pseudo-Boolean algebra (see [6]), whose presence of $[l, L]$ will have a strong influence upon the negation of images as well as upon the morphological operators. The main motivation is to establish a mathematical foundation using interval methods to introduce a new tool for mathematical morphology and that could be applied in processing images where it is required a more accuracy. Actually there are approach on interval fuzzy that is similar this interval images developed here, but the fuzzy theory, but is different in the sense of defining the elements based in the interval theory. the interval theory can found in [9, 10] and the fuzzy theory can be seen in [2, 8].

Algebraic structure of images with undefined information

Let $\Omega = \{[l, l], [l, L], [L, L]\}$, with $[l, l] \leq [l, L] \leq [L, L]$, where the operations $\vee$ and $\wedge$ are defined according to Table 1.

With the introduction of $[l, L]$ we observe that Ω is not a Boolean algebra, but a pseudo-Boolean algebra.

An operation that changes exact values, that is, changes the elements $[l, l]$ or $[L, L]$ to their dual values will be called **reversion**. We observe that the algebraic structure on Ω provides three reversion operations (see Table 2). Thus, $\langle \Omega, \bigvee, \bigwedge, [l, l], [L, L] \rangle$ is a complete pseudo-Boolean algebra, with three possible reversion operations: $\sim f$, $\neg f$, and $\overline{f}$.

It is not difficult to show that the struc-

Tabela 1: Binary operations $\vee$ and $\wedge$ on Ω

$\vee$	$[L,L]$	$[l,l]$	$[l,L]$
$[L,L]$	$[L,L]$	$[L,L]$	$[L,L]$
$[l,l]$	$[L,L]$	$[l,l]$	$[l,L]$
$[l,L]$	$[L,L]$	$[l,L]$	$[l,L]$

$\wedge$	$[L,L]$	$[l,l]$	$[l,L]$
$[L,L]$	$[L,L]$	$[l,l]$	$[l,L]$
$[l,l]$	$[l,l]$	$[l,l]$	$[l,l]$
$[l,L]$	$[l,L]$	$[l,l]$	$[l,L]$

ture $\langle \Omega, \vee, \wedge, [L,L], [l,l] \rangle$ is a distributive complete lattice with relative pseudo-complement. Table 2 shows the operations of negation, pseudo-complement ($\wedge$-complement) and $\vee$-complement on Ω.

Tabela 2: Negation, $\wedge$-complement, $\vee$-complement on Ω

$\sim$	$[l,l]$	$[l,L]$	$[L,L]$
	$[L,L]$	$[l,L]$	$[l,l]$

$\neg$	$[l,l]$	$[l,L]$	$[L,L]$
	$[L,L]$	$[l,l]$	$[l,l]$

$-$	$[l,l]$	$[l,L]$	$[L,L]$
	$[L,L]$	$[L,L]$	$[l,l]$

In the following way: value $[L,L]$ means white pixel; value $[l,l]$, black pixel and value $[l,L]$ the tone in that position is not clearly distinguished.

Interval Ternary Images

The following functions model images which pixels are that describe in the last section. They will be functions of the form $f : E \to \Omega$, where E is a finite set of coordinates and Ω is the previous lattice.We call those images **interval ternary images**.

Definition 1 *An undefined image is a function of the form $f : E \to \Omega$, where E is a finite set of coordinates and Ω is the previous lattice. The set of ternary images is denoted by Ω^E, and we can establish the following order relation on Ω^E: $f \leq g$ if, and only if, for any $x \in E$, $f(x) \leq g(x)$*

Definition 2 *Given a non-empty family of ternary images , $f_I = \{f_i\}_{i \in I}$, we can define the following basic operations:*

$$\bigvee\{f_i\} = \begin{cases} \text{[L,L]}, & se\ \exists i \in I, f_i(x) = [L,L] \\ \text{[l,l]}, & se\ \forall i \in I, f_i(x) = [l,l] \\ \text{[l,L]}, & otherwise \end{cases} \quad (1)$$

$$\bigwedge\{f_i\}(x) = \begin{cases} \text{[L,L]}, & se\ \forall i \in I, f_i(x) = [L,L] \\ \text{[l,l]}, & se\ \exists i \in I, f_i(x) = [l,l] \\ \text{[l,L]}, & otherwise \end{cases} \quad (2)$$

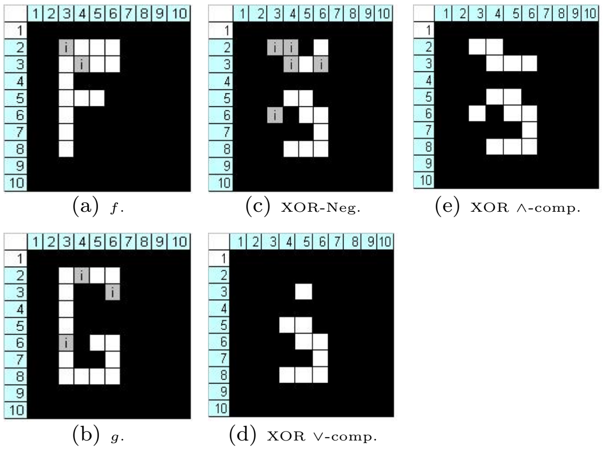
Proposition 1 *$\langle \Omega^E, \bigvee, \bigwedge, [l,l], [L,L] \rangle$ is a complete lattice.*

Corollary 1 (Pseudo-complement on Ω^E) *Given $f \in \Omega^E$, the **pseudo-complement** of f, denoted by $\neg f$, is given by expression "$\neg f = (f \Rightarrow \bot)$", where $\bot \in \Omega^E$ is the function, previously defined, $\bot(x) = [l,l]$.*

Definition 3 (Reversion operations) *An operation $\rho : \Omega^E \to \Omega^E$, is said to be a **reversion operation**, if, for any $f \in \Omega^E$ and for any $x \in E$, $\rho(f)(x) = y$ where $y \wedge x = [l,l]$ and $y \vee x = [L,L]$, every time that $f(x) \neq [l,L]$.*

Corollary 2 *The operations $\sim f$, $\neg f$ e $\overline{f}$ are **reversion operation** on Ω^E.*

Some remarks In this section, we showed that the space of interval ternary images is not a Boolean algebra, but a weaker algebraic space, namely pseudo-boolean algebra. Ordinary operations such as supremum and infimum plus three reversion operations have been exposed. It is easy to note that the usual complement on boolean algebras is a reversion. The proposed reversions differ only in the treatment of inaccuracy. Obviously, since the inaccuracy does not occur in the binary case, all reversion operations coincide with complement. The kinds of reversions above-mentioned can be interpreted in the following way: pseudo-complement as much as the $\vee$-complement are operations which deal with inaccuracy in an ad-hoc way, while negation leaves to the user (human or not) the decision about the detected inaccuracy. It is easy to note that Reversion operations. Will affect any operation XOR which will the composed of them. In what follows we show the impact of inaccuracy on morphological operations. The following figure 1 shows the operation defined with the provides reversion operations between two ternary images f and g, namely $XOR_{\neg} = (f \wedge \neg g) \vee (g \wedge \neg f)$; $XOR_{\sim} = (f \wedge \sim g) \vee (g \wedge \sim f)$ and $\overline{XOR} = (f \wedge \overline{g}) \vee (g \wedge \overline{f})$



(a) f. (c) XOR-Neg. (e) XOR $\wedge$-comp. (b) g. (d) XOR $\vee$-comp.

Figura 1: Reversion operations XOR for interval ternary images between f and g

Mathematics Morphology for undefined images

The theory of mathematical morphology is based on lattice theory [4, 3, 5] and aims the topological transformations of images. In this section we will define the elementary morphological operators on ternary images. To define the morphological operators we need to introduce the concepts of translation and Minkowski operations. Notice that our structuring elements are also ternary images. Given two ternary images f_A and f_B, let $B = \{u \in E : f_B(u) \neq [l, l]\}$. The **translation of f_A by a not null coordinate of f_B**, $u \in B$, is the function $f_{A+u} : E \to \Omega$, where:

$$f_{A+u}(x) = f_A(x-u)\,.$$

The Minkowski addition and subtraction are defined respectively by

$$f_A \oplus f_B = \bigvee_{u\in B} f_{A+u}\,, \tag{3}$$

$$f_A \ominus f_B = \bigwedge_{u\in B} f_{A-u}\,. \tag{4}$$

We know that an operator is a **dilation**, if for all family of images $\{f_i\} \subseteq \mathcal{T}$, $\psi(\sup\{f_i\}) = \sup\psi(\{f_i\})$, and it is an **erosion**, if $\psi(\inf\{f_i\}) = \inf\psi(\{f_i\})$.

Definition 4 *The **dilation** and the **erosion** of f_A with respect to B, denoted respectively by $\delta_B(f_A)$ and $\varepsilon_B(f_A)$, are defined by:*

$$\delta_B(f_A)(x) = \bigvee_{u\in B} f_A(x-u)$$

and

$$\varepsilon_B(f_A)(x) = \bigwedge_{u\in B} f_A(x+u)\,.$$

In other words, $\delta_B(f_A)(x) = (f_A \oplus f_B)(x)$ and $\varepsilon_B(f_A)(x) = (f_A \ominus f_B)(x)$, where the ternary image f_B is a structuring element.

Proposition 2 *The functions $\delta_B, \varepsilon_B : \mathcal{T} \to \mathcal{T}$ are a dilation and an erosion, respectively.*

Proof 1 *$\bigvee \delta_B(\{f_i\}_{i\in I})(x) = \bigvee \delta_B(\{f_A : f_A \in \{f_i\}_{i\in I}\})(x) = \bigvee\{\delta_B(f_A) : f_A \in \{f_i\}_{i\in I}\}(x) = \bigvee\{\delta_B(f_A)(x) : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_{A+u} : u \in B\}(x) : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_{A+u}(x) : u \in B\} : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_A(x-u) : u \in B\} : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_A(x-u) : f_A \in \{f_i\}_{i\in I}\} : u \in B\}$ (by associativity and commutativity of "$\bigvee$"). Moreover, making $h_A = \bigvee\{f_i\}_{i\in I}$, $\delta_B(h_A)(x) = \bigvee_{u\in B}\{h_{A+u}\}(x) = \bigvee\{h_{A+u} : u \in B\}(x) = \bigvee\{h_{A+u}(x) : u \in B\} = \bigvee\{h_A(x-u) : u \in B\} = \bigvee\{\bigvee\{f_i\}_{i\in I}(x-u) : u \in B\} = \bigvee\{\bigvee\{f_A : f_A \in \{f_i\}_{i\in I}\}(x-u) : u \in B\} = \bigvee\{\bigvee\{f_A(x-u) : f_A \in \{f_i\}_{i\in I}\} : u \in B\}$. Therefore, for all x, $\bigvee \delta_B(\{f_i\}_{i\in I})(x) = \delta_B(\bigvee\{f_i\}_{i\in I})(x)$; i.e., $\bigvee \delta_B(\{f_i\}_{i\in I}) = \delta_B(\bigvee\{f_i\}_{i\in I})$.*

By duality we prove that ε_B is an erosion.

Two important morphological operators are **openings** and **closings**, commonly used for *noise filtering*. They can be defined as compositions of erosions and dilations, with an adequate reversion operation.

Example 1 *The figure 2 shows the dilatation, erosion, of f_A with respect to f_B, where i denote the interval $[l, L]$. The figure 3 shows, one example the closing and opening operations. And The figure 4 shows, one example the morphological gradient that is an operations that involves subtraction between images and therefore we have three complement operations.*

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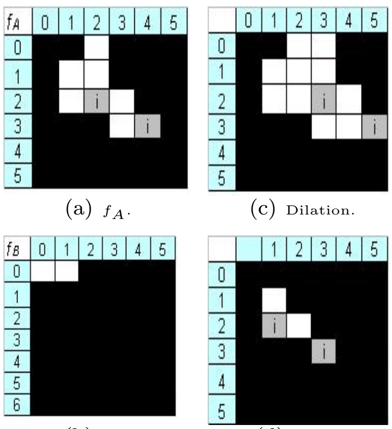
Final Remarks

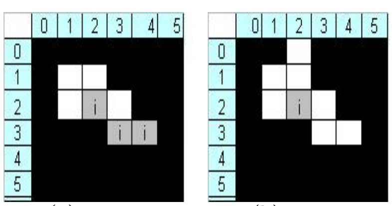
This work presented an algebraic structure and morphological operations for interval ternary images which include uncertainty with respect to the tone of some coordinates. We propose a algebraic structure that presents the notion of uncertainty, caught through of the interval, between two consecutive gray tones. The resulting algebraic structure is a



(a) f_A. (c) Dilation.

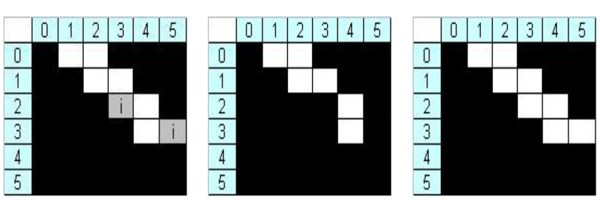
(b) f_B. (d) Erosion.

Figura 2: Dilation and erosion operation for ternary images



(a) Opening. (b) Closing.

Figura 3: Closing and Opening Operations for ternary images



(a) Grad Neg. (b) Grad $\vee$-comp. (c) Grad $\wedge$-comp.

Figura 4: Gradient Morphological for ternary images

pseudo-boolean algebra without complement operation. The main consequence for that is the presence of the so called reversion operations. However, reversions are not the only consequence of uncertainty, another one is the influence of them on translations and hence in all other morphological operations. The approach here is only a particular case of a more general theory of gray scale images and is being developed currently. Such idea belong to the notion of interval image that can see in [1]

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