On the Solution of Poisson's Equation in terms of the Green's Function Expanded in Eigenfunctions

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Abstract: In this work, we discuss some aspects of the use of the Green's functions expanded in eigenfunctions. To this end, we completely solve a Dirichlet problem based on Poisson's equation in plane polar coordinates. The problem geometry and boundary conditions were chosen so as to show up the features investigated, including an uncommon eigenvalue problem. We make an attempt to fill up a gap in the literature with the systematic and detailed calculation presented here.

Keywords: Poisson, Green, eigenfunction.

1) Introduction

Green's functions are often determined as an expansion in eigenfunctions. Depending on the number of spatial dimensions, two or more sets of orthogonal eigenfunctions may become available. An issue which surely arises is that of deciding which of the sets is the more appropriate. This question is addressed in this work. It is shown that the choice of the eigenfunctions is of special importance for problems with non-homogeneous boundary conditions.

To this end, we solve Poisson's equation in a half-disk under non-homogeneous Dirichlet boundary conditions employing, obviously, the plane polar coordinates. This problem also enables the discussion of several features concerning the Green's function method. In special, an uncommon eigenvalue problem is encountered, whose spectrum is continuous, in spite of the fact that the problem domain is bounded.

Section 2 contains the formulation of the problem considered as well as of its solution in terms of Green's function. Section 3 presents the calculation of Green's function as expansions in two kinds of eigenfunctions. Section 4 describes the determination of the problem solution in terms of the Green's functions calculated in Section 3. Section 5 concludes the body of the paper with a discussion of the results.

2) Formulation of the Problem and of the Solution

It is well established (References [1, Sec. 12.8], [3, Sec. 11.9], [4, Sec. 1.10] and [5, Sec. 7.2]) that the solution of Poisson's equation under the Dirichlet boundary condition,

$$\nabla^2\psi(\vec{\rho}) = -4\pi f(\vec{\rho}) \quad [\,\vec{\rho}\in\mathcal{A}\,] \ , \tag{1}$$

$$\psi(\vec{\rho}) = g(\vec{\rho}) \quad [\,\vec{\rho}\in\partial\mathcal{A}\,] \ , \tag{2}$$

where $\mathcal{A}$ is a domain of $\mathbb{R}^2$ and $\partial\mathcal{A}$ is its boundary, is given by

$$\psi(\vec{\rho}) = \int_{\mathcal{A}} dA'\, G(\vec{\rho}\,|\,\vec{\rho}\,')\, f(\vec{\rho}\,') - \frac{1}{4\pi}\oint_{\partial\mathcal{A}} ds'\, \frac{\partial G}{\partial n'}(\vec{\rho}\,|\,\vec{\rho}\,')\, g(\vec{\rho}\,') \ , \tag{3}$$

where $G(\vec{\rho}|\vec{\rho}\,')$ $[\,\vec{\rho}\,'\in\mathcal{A}\,]$ is the solution of

$$\nabla^2 G(\vec{\rho}\,|\,\vec{\rho}\,') = -4\pi\,\delta(\vec{\rho}-\vec{\rho}\,') \quad [\,\vec{\rho}\in\mathcal{A}\,] \ , \tag{4}$$

$$G(\vec{\rho}\,|\,\vec{\rho}\,') = 0 \quad [\,\vec{\rho}\in\partial\mathcal{A}\,] \ , \tag{5}$$

being $\partial G/\partial n$ the normal derivative, equal to $\vec{n}\cdot\nabla G$, with the unit normal vector $\vec{n}$ directed outward from $\mathcal{A}$ at a point of $\partial\mathcal{A}$.

Let $\mathcal{A}$ be the half-disk shown in Figure 1 below. In this geometry, the plane polar coordinates ρ and φ are the most suitable. For the boundary conditions also given in that figure, (1) and (2) become

$$\nabla^2\psi(\rho,\varphi) = -4\pi f(\rho,\varphi) \ , \tag{6}$$

$$\psi(1,\varphi) = g_1(\varphi) \ , \tag{7}$$

$$\psi(\rho,0) = 0 \ , \qquad \psi(\rho,\pi) = g_\pi(\varphi) \ , \tag{8}$$

with $0 \le \rho \le 1$ and $0 \le \varphi \le \pi$. Likewise, Equations (4) and (5) now read

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2}(\rho,\varphi \mid \rho',\varphi')$$
$$= -(4\pi/\rho)\,\delta(\rho-\rho')\,\delta(\varphi-\varphi') \ , \qquad (9)$$

$$G(1,\varphi \mid \rho',\varphi') = 0 \ , \qquad (10)$$

$$G(\rho,0 \mid \rho',\varphi') = G(\rho,\pi \mid \rho',\varphi') = 0 \ , \qquad (11)$$

with ρ' and φ' varying as ρ and φ vary. The solution of our problem [the problem defined by (6) to (8)] is, in accordance with (3),

$$\psi(\rho,\varphi) \equiv \psi_f(\rho,\varphi) + \psi_1(\rho,\varphi) + \psi_\pi(\rho,\varphi) \ , \qquad (12)$$

where the source term is given by

$$\psi_f = \int_0^\pi \int_0^1 G(\rho,\varphi \mid \rho',\varphi')\, f(\rho',\varphi')\, \rho' d\rho' d\varphi' \ , \qquad (13)$$

and the boundary terms, by

$$\psi_1(\rho,\varphi) = \frac{-1}{4\pi} \int_0^\pi \frac{\partial G}{\partial \rho'}(\rho,\varphi \mid 1,\varphi')\, g_1(\varphi')\, d\varphi' \ , \qquad (14)$$

$$\psi_\pi(\rho,\varphi) = \frac{-1}{4\pi} \int_0^1 \frac{\partial G}{\partial \varphi'}(\rho,\varphi \mid \rho',\pi)\, g_\pi(\rho') \frac{d\rho'}{\rho'} \ . \qquad (15)$$

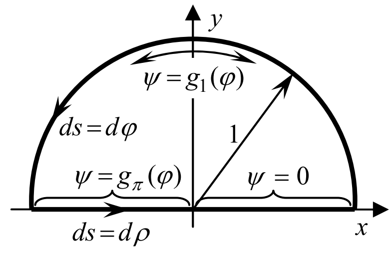

Figure 1 – Specification of the domain $\mathcal{A}$ and of the boundary data $g(\varphi)$ for the problem defined by (1) and (2).

3) Calculation of Green's Function

To calculate G, we consider two subregions of $\mathcal{A}$, those obtained with either a radial division, $\rho < \rho'$ and $\rho > \rho'$, or a sectorial division, $\varphi < \varphi'$ and $\varphi > \varphi'$ (Figure 2 below). Let us consider the radial division first. In each of them $\rho \ne \rho'$, and $\delta(\rho-\rho')$ vanishes, that is, the PDE for G given by (9) is homogeneous and can be solved by means of the method of separation of variables. Thus, substituting $G \equiv R(\rho)\,F(\varphi)$, we obtain

$$(\rho^2 R'' + \rho R')/R \ + \ \underbrace{(F''/F)}_{-\mu} = 0 \ , \qquad (16)$$

where we have recognized that the second term must be a constant, $-\mu$. The separated ODE $F'' + \mu F(\varphi) = 0$ is to be solved in each subregion under the boundary conditions $F(0) = F(\pi) = 0$ [derived from (11)]. This problem (with homogeneous ODE and boundary conditions) is an eigenvalue problem, whose solutions are easily found to be $\mu_n = n^2$ $(n = 1,2,3\cdots)$ and $F_n(\varphi) = \sin n\varphi$ (Reference [1, Sec. 8.2]). [Notice that the ODE for $R(\rho)$ which can be separated does not lead to an eigenvalue problem; in fact, on the common boundary at $\rho = \rho'$ of both subregions, *homogeneous* conditions cannot be derived!]

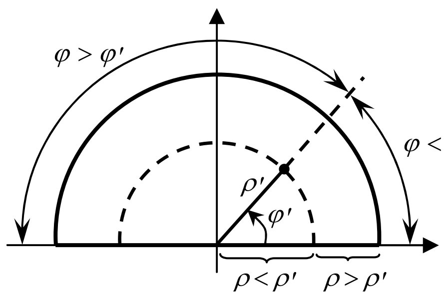

Figure 2 – The two ways the problem domain is divided in two subregions: radially ($\rho < \rho'$, $\rho > \rho'$) and sectorially ($\varphi < \varphi'$, $\varphi > \varphi'$).

The eigenfunctions above can be use to express Green's function as a linear superposition of terms of the type $R_n(\rho)F_n(\rho) = R_n(\rho)\sin n\varphi$:

$$G(\rho,\varphi \mid \rho',\varphi') = \sum_{n=1}^{\infty} R_n(\rho) \sin n\varphi \ . \qquad (17)$$

To determine the functions $R_n(\rho)$ (whose dependence on ρ' and φ' is implicit), we substitute the above expansion into the PDE given by (9), obtaining

$$\sum_{n=1}^{\infty} \left[R_n'' + \rho^{-1} R_n' - (n/\rho)^2 R_n(\rho) \right] \sin n\varphi$$

$$= -(4\pi/\rho)\,\delta(\rho-\rho')\,\delta(\varphi-\varphi') \ , \qquad (18)$$

from which we infer that the term enclosed by brackets are the coefficients of the Fourier sine series of the function on the right-hand side over the interval $(0,\pi)$, that is,

$$R_n'' + \rho^{-1} R_n' - (n/\rho)^2 R_n(\rho) =$$

$$\frac{2}{\pi}\int_0^{\pi} \left[\frac{-4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi') \right] \sin n\varphi \, d\varphi$$

$$= -(8\sin n\varphi')\,\delta(\rho-\rho')/\rho \ . \qquad (19)$$

This is an Euler equation, which is homogeneous in each half-ring (where $\rho \neq \rho'$). Its solution is (Reference [3, Sec. 1.6])

$$R_n(\rho) = \begin{cases} A_{1n}\,\rho^n + B_{1n}\rho^{-n} & (\rho < \rho') \\ A_{2n}\,\rho^n + B_{2n}\rho^{-n} & (\rho > \rho') \end{cases} \qquad (20)$$

The four constants above are determined by imposing the following four conditions (Reference [1, Sec. 12.2]):

(i) Finiteness at the origin, which is achieved by setting $B_{1n} = 0$.

(ii) The condition $R_n(1) = A_{2n} + B_{2n} = 0$ which follows from (10).

(iii) The continuity condition $R(\rho'^{+}) = R(\rho'^{-})$ at $\rho = \rho'$, since $G = RF$ is a potential and, therefore, must be a continuous function. Observe the notation $\rho'^{\pm} \equiv \rho' \pm \varepsilon$, with $\varepsilon \to 0^{+}$.

(iv) The jump discontinuity condition for the derivative of $R(\rho)$,

$$R'(\rho'^{+}) - R'(\rho'^{-}) = (-8\sin n\varphi')/\rho' ,$$

obtained by integrating (19) in the neighborhood of ρ' , from ρ'^{-} to ρ'^{+} .

Once the calculation of (20) is completed, its substitution into (17) yields

$$G^{\Cap}(\rho,\varphi \,|\, \rho',\varphi') =$$

$$\sum_{n=1}^{\infty} \frac{-4}{n} \rho_<^{n} (\rho_>^{n} - \rho_>^{-n}) \sin n\varphi' \sin n\varphi \ , \qquad (21)$$

where $\rho_<$ $(\rho_>)$ is the smaller (larger) of ρ and ρ' . We will use the symbol $\Cap$ to indicate the calculation of Green's function considering the radial division of $\mathcal{A}$.

Let us calculate Green's function again, this time considering the region $\mathcal{A}$ divided in the two sectors $\varphi < \varphi'$ and $\varphi > \varphi'$. In both of them, it is now for $R(\rho)$ that an eigenvalue problem arises with the separation of variables $G \equiv R(\rho)\,F(\varphi)$, because of the homogeneous condition $R(1) = 0$ [deduced from (10)] on the boundary at $\rho = 1$ of both sectors. We thus separate the ODE $\rho^2 R'' + \rho R' + \lambda R(\rho) = 0$ by equating the first term in (16) to the constant $-\lambda$.

The eigenvalue problem so obtained can be converted to a familiar one by changing the independent variable to $u \equiv -\ln\rho$. It becomes $\bar{R} + \lambda \bar{R}(u) = 0$, with $\bar{R}(0) = 0$ and $u \geq 0$, where $\bar{R}(u) \equiv R[\rho(u)]$ and $\rho(u) = e^{-u}$. The well known eigenvalues and eigenfunctions are $\lambda_k = k^2$ and $\bar{R}_k(u) = \sin ku$ [or $R_k(\rho) =$ $\sin(\ln\rho)$], with $k > 0$ (a continuous spectrum: cf. Reference [1, Sec. 8.7]). In many instances, it is better to work with the new variable u, in terms of which (9) reads

$$\frac{\partial^2 \bar{G}}{\partial u^2} + \frac{\partial^2 \bar{G}}{\partial \varphi^2} = -4\pi\,\delta(u-u')\,\delta(\varphi-\varphi') , \qquad (22)$$

where $\bar{G}(u,\varphi \,|\, u',\varphi') \equiv G[\rho(u),\varphi \,|\, \rho'(u'),\varphi']$.

The calculation of $\bar{G}$ proceeds in the same manner described above. We substitute the expansion

$$\bar{G}(u,\varphi \,|\, u',\varphi') = \int_0^{\infty} F_k(\varphi) \sin ku \, du \qquad (23)$$

into (22) to obtain

$$\int_0^{\infty} \left[F_k'' - k^2 F_k(\varphi) \right] \sin ku \, du$$

$$= -4\pi\,\delta(u-u')\,\delta(\varphi-\varphi') \ . \qquad .$$

Then, by using the Fourier sine integral formula (Reference [2, Sec. 64]), we calculate the term in the integrand which is enclosed by brackets, obtaining the equation

$$F_k'' - k^2 F_k(\varphi) =$$

$$\frac{2}{\pi}\int_0^\infty \left[-4\pi\,\delta(u-u')\,\delta(\varphi-\varphi')\right]\sin ku\,du$$

$$= -8\sin ku'\,\delta(\varphi-\varphi') \ . \qquad (24)$$

Next, we solve it separately in each sector,

$$F_k(\varphi) = \begin{cases} A_{1k}\cosh k\varphi + B_{1k}\sinh k\varphi & (\varphi < \varphi') \\ A_{2k}\cosh k\varphi + B_{2k}\sinh k\varphi & (\varphi > \varphi') \end{cases}$$

and determine the four constants imposing the four conditions: (i) $F_k(0) = 0$ and (ii) $F_k(\pi) = 0$ [both from the boundary condition (11)]; (iii) $F_k(\varphi'^{+}) = F_k(\varphi'^{-})$ (continuity at $\varphi = \varphi'$); (iv) $F_k{}'(\varphi'^{+}) - F_k{}'(\varphi'^{-}) = -8\sin ku'$ (jump discontinuity of $F_k'(\varphi)$ at $\varphi = \varphi'$, derived by integrating (24) in the neighborhood of φ', from φ'^{-} to φ'^{+}).

Finally, we substitute the $F_k(\varphi)$ so determined into (23):

$$\bar{G}^{\vee}(u,\varphi\,|\,u',\varphi') =$$

$$\int_0^\infty \frac{8\sin ku\sin ku'\sinh k\varphi_<\sinh k(\pi-\varphi_>)\,dk}{k\sinh k\pi} \ , \qquad (25)$$

where $\varphi_<$ $(\varphi_>)$ is the smaller (larger) of φ and φ'. The symbol $^{\vee}$, like a sector, indicates that G was calculated considering $\mathcal{A}$ divided in two sectors.

4) The Solution in terms of the Calculated Green's Function

We develop below only the boundary terms $\psi_1(\rho,\varphi)$ and $\psi_\pi(\rho,\varphi)$ in (12); the source term $\psi_f(\rho,\varphi)$ is pretty well discussed in the literature (e.g., Reference [4]).

Looking at (14) and (15), we see that we need to calculate $\partial G/\partial\rho'$ at $\rho' = 1$ and $\partial G/\partial\varphi'$ at $\varphi' = \pi$. Using (21) first, we obtain

$$\frac{\partial G^{\Cap}}{\partial\rho'}(\rho,\varphi\,|\,1,\varphi') =$$

$$\left.\frac{\partial}{\partial\rho'}\sum_{n=1}^\infty \frac{-4}{n}\rho^n(\rho'^{n} - \rho'^{-n})\sin n\varphi'\sin n\varphi\right|_{\rho'=1}$$

$$= -8\sum_{n=1}^\infty \rho^n\sin n\varphi'\sin n\varphi$$

and

$$\frac{\partial G^{\Cap}}{\partial\varphi'}(\rho,\varphi|\rho',\pi) =$$

$$\left.\frac{\partial}{\partial\varphi'}\sum_{n=1}^\infty \frac{-4}{n}\rho_<^n(\rho_>^n - \rho_>^{-n})\sin n\varphi'\sin n\varphi\right|_{\varphi'=\pi}$$

$$= -4\sum_{n=1}^\infty (-1)^n\rho_<^n(\rho_>^n - \rho_>^{-n})\sin n\varphi \ .$$

Substituting these results into (14) and (15) , we obtain

$$\psi_1^{\Cap}(\rho,\varphi) = \sum_{n=1}^\infty \gamma_{1n}\,\rho^n\sin n\varphi \ , \qquad (26)$$

$$\psi_\pi^{\Cap}(\rho,\varphi) \equiv \frac{1}{\pi}\sum_{n=1}^\infty (-1)^n I_n(\rho)\sin n\varphi \ , \qquad (27)$$

where

$$\gamma_{1n} \equiv \frac{2}{\pi}\int_0^\pi g_1(\varphi')\sin n\varphi'd\varphi' \ ,$$

$$I_n(\rho) \equiv \int_0^1 g_\pi(\rho')\rho_<^n(\rho_>^n - \rho_>^{-n})\,d\rho'/\rho' \ .$$

Now using (25) to calculate $\partial G/\partial\rho'$ at $\rho' = 1$ and $\partial G/\partial\varphi'$ at $\varphi' = \pi$, we get

$$\frac{\partial G^{\vee}}{\partial\rho'}(\rho,\varphi\,|\,1,\varphi') = \left[-e^{u'}\frac{\partial}{\partial u'}\int_0^\infty dk\right.$$

$$\left.\frac{8\sin ku\sin ku'\sinh k\varphi_<\sinh k(\pi-\varphi_>)}{k\sinh k\pi}\right]_{u'=0}$$

$$= -8\int_0^\infty \frac{\sin ku\sinh k\varphi_<\sinh k(\pi-\varphi_>)}{\sinh k\pi}dk \ .$$

and

$$\frac{\partial G^{\vee}}{\partial \varphi'}(\rho,\varphi\,|\,\rho',\pi) \;=\; \left[\frac{\partial}{\partial \varphi'}\int_0^{\infty} dk \right.$$

$$\left.\frac{8\sin ku \sin ku' \sinh k\varphi \sinh k(\pi-\varphi')}{k\,\sinh k\pi}\right]_{\varphi'=\pi}$$

$$= -8\int_0^{\infty} \frac{\sin ku \sin ku' \sinh k\varphi}{\sinh k\pi}\,dk \;.$$

Substitution into (14) and (15) gives

$$\psi_1^{\vee}(\rho,\varphi) \equiv \frac{-2}{\pi}\int_0^{\infty} dk \frac{\sin(k\ln\rho)}{\sinh k\pi} I_k(\varphi) \;, \qquad (28)$$

$$\psi_\pi^{\vee}(\rho,\varphi) \equiv -\int_0^{\infty} dk \frac{\sinh k\varphi}{\sinh k\pi}\gamma_\pi(k)\sin(k\ln\rho)\,, \qquad (29)$$

where

$$I_k(\varphi) \equiv \int_0^{\pi} d\varphi'\, g_1(\varphi')\sinh k\varphi_< \sinh k(\pi-\varphi_>)\,,$$

$$\gamma_\pi(k) \equiv \frac{2}{\pi}\int_0^{\infty} du'\,\bar{g}_\pi(u')\sin ku' \;.$$

5) Discussion of the Results

a) Notice that, using (26) and (29),

$$\psi_1^{\Cap}(1,\varphi) = \sum_{n=1}^{\infty} \gamma_{1n}\sin n\varphi = g_1(\varphi) \;,$$

$$\psi_\pi^{\vee}(\rho,\pi) = \int_0^{\infty} dk\,\gamma_\pi(k)\sin ku = g_\pi(\rho) \;;$$

but, using (28) and (27),

$$\psi_1^{\vee}(1,\varphi) \equiv \frac{-2}{\pi}\int_0^{\infty} dk \frac{\sin(k\ln 1)}{\sinh k\pi} I_k(\varphi) \equiv 0 \;,$$

$$\psi_\pi^{\Cap}(\rho,\pi) \equiv \frac{1}{4\pi}\sum_{n=1}^{\infty}(-1)^n \sin n\pi\; I_n(\rho) \equiv 0 \;.$$

Therefore, the part $\psi_1(\rho,\varphi)$ of the solution, due to the non-homogeneous boundary data $g_1(\varphi)$ [cf. (12) and (14)], *is better* given by $\psi_1^{\Cap}(\rho,\varphi)$ [which is built with the Green's function given by (21)], since it converges to that data, what does not happen with $\psi_1^{\vee}(\rho,\varphi)$. For a similar reason, the part $\psi_\pi(\rho,\varphi)$ of the solution *is better* given by $\psi_\pi^{\vee}(\rho,\varphi)$.

As a matter of fact, if the boundary data are function of some variable, Green's function "is better" expanded in the eigenfunctions which depend on that variable.

In the above, we just say "is better" instead of "must be", because both $\psi^{\Cap}$ and $\psi^{\vee}$ converge everywhere in the (open) domain $\mathcal{A}$, and ψ is *known* on the boundary $\partial\mathcal{A}$. Therefore, the fact that $\psi^{\Cap}$ $(=\psi_1^{\Cap} + \psi_\pi^{\Cap})$ cannot reproduce the boundary data $g_\pi(\varphi)$ and $\psi^{\vee}$ $(=\psi_1^{\vee}+\psi_\pi^{\vee})$ cannot reproduce the data $g_1(\rho)$ would be of no consequence if it were not a corollary that the convergence of $\psi_\pi^{\Cap}$ and $\psi_1^{\vee}$ in $\mathcal{A}$ will be more difficult to achieve than that of $\psi_\pi^{\vee}$ and $\psi_1^{\Cap}$, respectively.

b) The Green's function $G^{\Cap}(\rho,\varphi\,|\,\rho',\varphi)$ can be expressed in closed form. In fact, simplifying the notation by defining $p\equiv\rho_<\rho_> = \rho'\rho$, $q\equiv\rho_</\rho_>$, $d\equiv\varphi'-\varphi$ and $s\equiv\varphi'+\varphi$, we can develop (21) as follows:

$$G^{\Cap}(\rho,\varphi\,|\,\rho',\varphi) =$$

$$= -2\sum_{n=1}^{\infty}(p^n/n)\cos nd + 2\sum_{n=1}^{\infty}(q^n/n)\cos nd$$

$$+\,2\sum_{n=1}^{\infty}(p^n/n)\cos ns - 2\sum_{n=1}^{\infty}(q^n/n)\cos ns \;. \qquad (30)$$

However, notice that

$$-2\sum_{n=1}^{\infty}(r^n/n)\cos n\theta \;=\; -2\,\mathrm{Re}\sum_{n=1}^{\infty}(z^n/n)$$

$$= 2\,\mathrm{Re}\left\{\log(1-z)\right\} \;=\; 2\ln|1-z|$$

$$= \ln(1-2r\cos\theta + r^2) \;, \qquad (31)$$

where $z = r\,e^{\mathrm{i}\theta}$, and the well known Taylor's series of $\log(1-z)$ was used. (In the above, we distinguish between the complex logarithmic function and the real one by employing the notations log and ln, respectively). We can, therefore, use the formula in (31) to replace each series in (30) by a logarithmic term, thus accomplishing our intent of expressing Green's function in a closed form:

$$G^{\Cap}(\rho,\varphi \mid \rho',\varphi) =$$

$$= \ln(1-2p\cos d + p^2) - \ln(1-2q\cos d + q^2)$$

$$-\ln(1-2p\cos s + p^2) + \ln(1-2q\cos s + q^2) \ .$$

c) The integral which furnishes the Green's function $\bar{G}^{\vee}(u,\varphi \mid u',\varphi')$, in (25), can be evaluated by considering it along the closed contour of the k-plane shown in Figure 3 below (where the radius tends to infinite: $R \to \infty$). It is a simple matter to show that

$$\bar{G}^{\vee}(u,\varphi \mid u',\varphi') = (1/2)\,2\pi\mathrm{i}\sum_{n=1}^{\infty}\mathrm{Res}\,(n\mathrm{i}) =$$

$$-\sum_{n=1}^{\infty}\ (8/n)\sinh nu\,\sinh nu'\,\sin n\varphi_{<}\ \sin n\varphi_{>} \ .$$

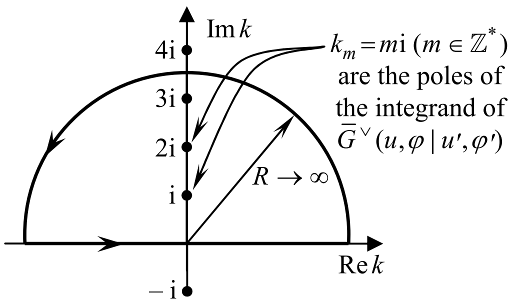

Figure 3 – The closed contour used to evaluate the real integral in (25) with the help of the residue theorem.

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