

SSJ User's Guide

Package `probdistmulti`

Multivariate Probability Distributions

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This package provides tools to compute densities, mass functions, distribution functions for various continuous and discrete multivariate probability distributions.

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Overview

This package contains Java classes providing methods to compute mass, density, distribution and complementary distribution functions for some multi-dimensional discrete and continuous probability distributions. It does not generate random numbers for multivariate distributions; for that, see the package `randvarmulti`.

Distributions

We recall that the *distribution function* of a *continuous* random vector $X = \{x_1, x_2, \dots, x_d\}$ with *density* $f(x_1, x_2, \dots, x_d)$ over the d -dimensional space R^d is

$$F(x_1, x_2, \dots, x_d) = P[X_1 \leq x_1, X_2 \leq x_2, \dots, X_d \leq x_d] \quad (1)$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_d} f(s_1, s_2, \dots, s_d) ds_1 ds_2 \dots ds_d \quad (2)$$

while that of a *discrete* random vector X with *mass function* $\{p_1, p_2, \dots, p_d\}$ over a fixed set of real numbers is

$$F(x_1, x_2, \dots, x_d) = P[X_1 \leq x_1, X_2 \leq x_2, \dots, X_d \leq x_d] \quad (3)$$

$$= \sum_{i_1 \leq x_1} \sum_{i_2 \leq x_2} \cdots \sum_{i_d \leq x_d} p(x_1, x_2, \dots, x_d), \quad (4)$$

where $p(x_1, x_2, \dots, x_d) = P[X_1 = x_1, X_2 = x_2, \dots, X_d = x_d]$. For a discrete distribution over the set of integers, one has

$$F(x_1, x_2, \dots, x_d) = P[X_1 \leq x_1, X_2 \leq x_2, \dots, X_d \leq x_d] \quad (5)$$

$$= \sum_{s_1=-\infty}^{x_1} \sum_{s_2=-\infty}^{x_2} \cdots \sum_{s_d=-\infty}^{x_d} p(s_1, s_2, \dots, s_d), \quad (6)$$

where $p(s_1, s_2, \dots, s_d) = P[X_1 = s_1, X_2 = s_2, \dots, X_d = s_d]$.

We define \bar{F} , the *complementary distribution function* of X , as

$$\bar{F}(x_1, x_2, \dots, x_d) = P[X_1 \geq x_1, X_2 \geq x_2, \dots, X_d \geq x_d]. \quad (7)$$

DiscreteDistributionIntMulti

Classes implementing multi-dimensional discrete distributions over the integers should inherit from this class. It specifies the signature of methods for computing the mass function (or probability) $p(x_1, x_2, \dots, x_d) = P[X_1 = x_1, X_2 = x_2, \dots, X_d = x_d]$ and the cumulative probabilities for a random vector X with a discrete distribution over the integers.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public abstract class DiscreteDistributionIntMulti
```

```
    public abstract double prob (int[] x);
```

Returns the probability mass function $p(x_1, x_2, \dots, x_d)$, which should be a real number in $[0, 1]$.

```
    public double cdf (int x[])
```

Computes the cumulative probability function F of the distribution evaluated at \mathbf{x} , assuming the lowest values start at 0, i.e. computes

$$F(x_1, x_2, \dots, x_d) = \sum_{s_1=0}^{x_1} \sum_{s_2=0}^{x_2} \cdots \sum_{s_d=0}^{x_d} p(s_1, s_2, \dots, s_d).$$

Uses the naive implementation, is very inefficient and may underflows.

```
    public int getDimension()
```

Returns the dimension d of the distribution.

```
    public abstract double[] getMean();
```

Returns the mean vector of the distribution, defined as $\mu_i = E[X_i]$.

```
    public abstract double[][] getCovariance();
```

Returns the variance-covariance matrix of the distribution, defined as $\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$.

```
    public abstract double[][] getCorrelation();
```

Returns the correlation matrix of the distribution, defined as $\rho_{ij} = \sigma_{ij} / \sqrt{\sigma_{ii}\sigma_{jj}}$.

ContinuousDistributionMulti

Classes implementing continuous multi-dimensional distributions should inherit from this class. Such distributions are characterized by a *density* function $f(x_1, x_2, \dots, x_d)$; thus the signature of a **density** method is supplied here. All array indices start at 0.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public abstract class ContinuousDistributionMulti
```

```
    public abstract double density (double[] x);
```

Returns $f(x_1, x_2, \dots, x_d)$, the probability density of X evaluated at the point x , where $x = \{x_1, x_2, \dots, x_d\}$. The convention is that $x[i-1] = x_i$.

```
    public int getDimension()
```

Returns the dimension d of the distribution.

```
    public abstract double[] getMean();
```

Returns the mean vector of the distribution, defined as $\mu_i = E[X_i]$.

```
    public abstract double[][] getCovariance();
```

Returns the variance-covariance matrix of the distribution, defined as $\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$.

```
    public abstract double[][] getCorrelation();
```

Returns the correlation matrix of the distribution, defined as $\rho_{ij} = \sigma_{ij} / \sqrt{\sigma_{ii}\sigma_{jj}}$.

ContinuousDistribution2Dim

Classes implementing 2-dimensional continuous distributions should inherit from this class. Such distributions are characterized by a *density* function $f(x, y)$; thus the signature of a `density` method is supplied here. This class also provides a default implementation of $\bar{F}(x, y)$, the upper CDF. The inverse function $F^{-1}(u)$ represents a curve $y = h(x)$ of constant u and it is not implemented.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public abstract class ContinuousDistribution2Dim
    extends ContinuousDistributionMulti
```

```
public int decPrec = 15;
```

Defines the target number of decimals of accuracy when approximating a distribution function, but there is *no guarantee* that this target is always attained.

```
public abstract double density (double x, double y);
```

Returns $f(x, y)$, the density of (X, Y) evaluated at (x, y) .

```
public double density (double[] x)
```

Simply calls `density (x[0], x[1])`.

```
public abstract double cdf (double x, double y);
```

Computes the distribution function $F(x, y)$:

$$F(x, y) = P[X \leq x, Y \leq y] = \int_{-\infty}^x ds \int_{-\infty}^y dt f(s, t). \quad (8)$$

```
public double barF (double x, double y)
```

Computes the upper cumulative distribution function $\bar{F}(x, y)$:

$$\bar{F}(x, y) = P[X \geq x, Y \geq y] = \int_x^{\infty} ds \int_y^{\infty} dt f(s, t). \quad (9)$$

```
public double cdf (double a1, double a2, double b1, double b2)
```

Computes the cumulative probability in the square region

$$P[a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2] = \int_{a_1}^{b_1} dx \int_{a_2}^{b_2} dy f(x, y). \quad (10)$$

MultinomialDist

Implements the abstract class `DiscreteDistributionIntMulti` for the *multinomial* distribution with parameters n and (p_1, \dots, p_d) . The probability mass function is [5]

$$P[X = (x_1, \dots, x_d)] = \frac{n!}{\prod_{i=1}^d x_i!} \prod_{i=1}^d p_i^{x_i}, \quad (11)$$

where $\sum_{i=1}^d x_i = n$ and $\sum_{i=1}^d p_i = 1$.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class MultinomialDist extends DiscreteDistributionIntMulti
```

Constructors

```
public MultinomialDist (int n, double p[])
```

Creates a `MultinomialDist` object with parameters n and (p_1, \dots, p_d) such that $\sum_{i=1}^d p_i = 1$. We have $p_i = p[i-1]$.

Methods

```
public static double prob (int n, double p[], int x[])
```

Computes the probability mass function (11) of the multinomial distribution with parameters n and (p_1, \dots, p_d) evaluated at x .

```
public static double cdf (int n, double p[], int x[])
```

Computes the function F of the multinomial distribution with parameters n and (p_1, \dots, p_d) evaluated at x .

```
public static double[] getMean (int n, double[] p)
```

Computes the mean $E[X_i] = np_i$ of the multinomial distribution with parameters n and (p_1, \dots, p_d) .

```
public static double[][] getCovariance (int n, double[] p)
```

Computes the covariance matrix of the multinomial distribution with parameters n and (p_1, \dots, p_d) .

```
public static double[][] getCorrelation (int n, double[] p)
```

Computes the correlation matrix of the multinomial distribution with parameters n and (p_1, \dots, p_d) .

```
public static double[] getMaximumLikelihoodEstimate (int x[][], int m,  
                                                    int d, int n)
```

Estimates and returns the parameters $[\hat{p}_1, \dots, \hat{p}_d]$ of the multinomial distribution using the maximum likelihood method based on the m observations of d components in table $x[i][j]$, $i = 0, 1, \dots, m - 1$ and $j = 0, 1, \dots, d - 1$.

The equations of the maximum likelihood are defined as

$$\hat{p}_i = \frac{\bar{X}_i}{N}.$$

```
public int getN()
```

Returns the parameter n of this object.

```
public double[] getP()
```

Returns the parameters (p_1, \dots, p_d) of this object.

```
public void setParams (int n, double p[])
```

Sets the parameters n and (p_1, \dots, p_d) of this object.

NegativeMultinomialDist

Implements the abstract class `DiscreteDistributionIntMulti` for the *negative multinomial* distribution with parameters $\gamma > 0$ and (p_1, \dots, p_d) . The probability mass function is [5]

$$P[X = (x_1, \dots, x_d)] = \frac{\Gamma\left(\gamma + \sum_{i=1}^d x_i\right) p_0^{\gamma} \prod_{i=1}^d p_i^{x_i}}{\Gamma(\gamma) \prod_{i=1}^d x_i!} \quad (12)$$

where $\sum_{i=1}^d p_i < 1$, $p_0 = 1 - \sum_{i=1}^d p_i$ and $p_i = \lambda_i / \left(1 + \sum_{i=1}^d \lambda_i\right)$.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class NegativeMultinomialDist extends DiscreteDistributionIntMulti
```

Constructors

```
public NegativeMultinomialDist (double gamma, double p[])
```

Creates a `NegativeMultinomialDist` object with parameters $\gamma = \text{gamma}$ and (p_1, \dots, p_d) such that $\sum_{i=1}^d p_i < 1$, as described above. We have $p_i = \text{p}[i-1]$.

Methods

```
public static double prob (double gamma, double p[], int x[])
```

Computes the probability mass function (12) of the negative multinomial distribution with parameters γ and (p_1, \dots, p_d) , evaluated at x .

```
public static double cdf (double gamma, double p[], int x[])
```

Computes the cumulative probability function F of the negative multinomial distribution with parameters γ and (p_1, \dots, p_k) , evaluated at x .

```
public static double[] getMean (double gamma, double p[])
```

Computes the mean $E[X] = \gamma p_i / p_0$ of the negative multinomial distribution with parameters γ and (p_1, \dots, p_d) .

```
public static double[][] getCovariance (double gamma, double p[])
```

Computes the covariance matrix of the negative multinomial distribution with parameters γ and (p_1, \dots, p_d) .

```
public static double[][] getCorrelation (double gamma, double[] p)
```

Computes the correlation matrix of the negative multinomial distribution with parameters γ and (p_1, \dots, p_d) .

```
public static double[] getMaximumLikelihoodEstimate (int x[][] , int n,
                                                    int d)
```

Estimates the parameters $[\hat{\gamma}, \hat{p}_1, \dots, \hat{p}_d]$ of the negative multinomial distribution using the maximum likelihood method based on the n observations of d components in table $x[i][j]$, $i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, d-1$.

The equations of the maximum likelihood are defined in [5]:

$$\sum_{l=1}^M \frac{F_l}{(\hat{\gamma} + l - 1)} = \ln \left(1 + \frac{1}{n\hat{\gamma}} \sum_{j=1}^n \Upsilon_j \right)$$

$$p_i = \frac{\lambda_i}{1 + \sum_{j=1}^d \lambda_j} \quad \text{for } i = 1, \dots, d$$

where

$$\lambda_i = \frac{\sum_{j=1}^n X_{i,j}}{n\hat{\gamma}} \quad \text{for } i = 1, \dots, d$$

$$\Upsilon_j = \sum_{i=1}^d X_{i,j} \quad \text{for } j = 1, \dots, n$$

$$F_l = \frac{1}{n} \sum_{j=1}^n \mathbf{1}\{\Upsilon_j \geq l\} \quad \text{for } l = 1, \dots, M$$

$$M = \max_j \{\Upsilon_j\}$$

```
public double getGamma()
```

Returns the parameter γ of this object.

```
public double[] getP()
```

Returns the parameters (p_1, \dots, p_d) of this object.

```
public void setParams (double gamma, double p[])
```

Sets the parameters γ and (p_1, \dots, p_d) of this object.

BiNormalDist

Extends the class `ContinuousDistribution2Dim` for the *bivariate normal* distribution [6, page 84]. It has means $E[X] = \mu_1$, $E[Y] = \mu_2$, and variances $\text{var}[X] = \sigma_1^2$, $\text{var}[Y] = \sigma_2^2$ such that $\sigma_1 > 0$ and $\sigma_2 > 0$. The correlation between X and Y is ρ . Its density function is

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-T} \quad (13)$$

$$T = \frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x-\mu_1}{\sigma_1} \right) \left(\frac{y-\mu_2}{\sigma_2} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right)^2 \right]$$

and the corresponding distribution function is (the `cdf` method)

$$\Phi(\mu_1, \sigma_1, x, \mu_2, \sigma_2, y, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^x dx \int_{-\infty}^y dy e^{-T}. \quad (14)$$

We also define the upper distribution function (the `barF` method) as

$$\bar{\Phi}(\mu_1, \sigma_1, x, \mu_2, \sigma_2, y, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_x^{\infty} dx \int_y^{\infty} dy e^{-T}. \quad (15)$$

When $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, we have the *standard binormal* distribution, with corresponding distribution function

$$\Phi(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^x dx \int_{-\infty}^y dy e^{-S} \quad (16)$$

$$S = \frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}.$$

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class BiNormalDist extends ContinuousDistribution2Dim
```

Constructors

```
public BiNormalDist (double rho)
```

Constructs a `BiNormalDist` object with default parameters $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and correlation $\rho = \text{rho}$.

```
public BiNormalDist (double mu1, double sigma1,
                    double mu2, double sigma2, double rho)
```

Constructs a `BiNormalDist` object with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$ and $\rho = \text{rho}$.

Methods

```
public static double density (double x, double y, double rho)
```

Computes the *standard binormal* density function (13) with $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.

```
public static double density (double mu1, double sigma1, double x,
                             double mu2, double sigma2, double y,
                             double rho)
```

Computes the *binormal* density function (13) with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$ and $\rho = \text{rho}$.

```
public static double cdf (double x, double y, double rho)
```

Computes the standard *binormal* distribution (16) using the fast Drezner-Wesolowsky method described in [3]. The absolute error is expected to be smaller than $2 \cdot 10^{-7}$.

```
public static double cdf (double mu1, double sigma1, double x,
                          double mu2, double sigma2, double y,
                          double rho)
```

Computes the *binormal* distribution function (14) with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$ and $\rho = \text{rho}$. Uses the fast Drezner-Wesolowsky method described in [3]. The absolute error is expected to be smaller than $2 \cdot 10^{-7}$.

```
public static double barF (double x, double y, double rho)
```

Computes the standard upper *binormal* distribution with $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Uses the fast Drezner-Wesolowsky method described in [3]. The absolute error is expected to be smaller than $2 \cdot 10^{-7}$.

```
public static double barF (double mu1, double sigma1, double x,
                          double mu2, double sigma2, double y,
                          double rho)
```

Computes the upper *binormal* distribution function (15) with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$ and $\rho = \text{rho}$. Uses the fast Drezner-Wesolowsky method described in [3]. The absolute error is expected to be smaller than $2 \cdot 10^{-7}$.

```
public static double[] getMean(double mu1, double sigma1,
                              double mu2, double sigma2, double rho)
```

Return the mean vector $E[X] = (\mu_1, \mu_2)$ of the binormal distribution.

```
public static double[][] getCovariance (double mu1, double sigma1,
                                       double mu2, double sigma2,
                                       double rho)
```

Return the covariance matrix of the binormal distribution.

```
public static double[][] getCorrelation (double mu1, double sigma1,
                                       double mu2, double sigma2,
                                       double rho)
```

Return the correlation matrix of the binormal distribution.

```
public double getMu1()
```

Returns the parameter μ_1 .

```
public double getMu2()
```

Returns the parameter μ_2 .

```
public double getSigma1()
```

Returns the parameter σ_1 .

```
public double getSigma2()
```

Returns the parameter σ_2 .

```
protected void setParams (double mu1, double sigma1,  
                           double mu2, double sigma2, double rho)
```

Sets the parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$ and $\rho = \text{rho}$ of this object.

BiNormalGenzDist

Extends the class `BiNormalDist` for the *bivariate normal* distribution [6, page 84] using Genz's algorithm as described in [4].

```
package umontreal.iro.lecuyer.probdistmulti;  
  
public class BiNormalGenzDist extends BiNormalDist
```

Constructors

```
public BiNormalGenzDist (double rho)  
    Constructs a BiNormalGenzDist object with default parameters  $\mu_1 = \mu_2 = 0$ ,  $\sigma_1 = \sigma_2 = 1$   
    and correlation  $\rho = \text{rho}$ .  
  
public BiNormalGenzDist (double mu1, double sigma1,  
                        double mu2, double sigma2, double rho)  
    Constructs a BiNormalGenzDist object with parameters  $\mu_1 = \text{mu1}$ ,  $\mu_2 = \text{mu2}$ ,  $\sigma_1 = \text{sigma1}$ ,  
     $\sigma_2 = \text{sigma2}$  and  $\rho = \text{rho}$ .
```

Methods

```
public static double cdf (double x, double y, double rho)  
    Computes the standard binormal distribution (16) with the method described in [4]. The  
    code for the cdf was translated directly from the Matlab code written by Alan Genz and  
    available from his web page at http://www.math.wsu.edu/faculty/genz/homepage (the  
    code is copyrighted by Alan Genz and is included in this package with the kind permission  
    of the author). The absolute error is expected to be smaller than  $0.5 \cdot 10^{-15}$ .
```

BiNormalDonnellyDist

Extends the class BiNormalDist for the *bivariate normal* distribution [6, page 84] using a translation of Donnelly's FORTRAN code in [2].

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class BiNormalDonnellyDist extends BiNormalDist
```

Constructors

```
public BiNormalDonnellyDist (double rho, int ndig)
```

Constructor with default parameters $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, correlation $\rho = \text{rho}$, and $d = \text{ndig}$ digits of accuracy (the absolute error is smaller than 10^{-d}). Restriction: $d \leq 15$.

```
public BiNormalDonnellyDist (double rho)
```

Same as BiNormalDonnellyDist (rho, 15).

```
public BiNormalDonnellyDist (double mu1, double sigma1, double mu2,
                             double sigma2, double rho, int ndig)
```

Constructor with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$, $\rho = \text{rho}$, and $d = \text{ndig}$ digits of accuracy. Restriction: $d \leq 15$.

```
public BiNormalDonnellyDist (double mu1, double sigma1, double mu2,
                             double sigma2, double rho)
```

Same as BiNormalDonnellyDist (mu1, sigma1, mu2, sigma2, rho, 15).

Methods

The following methods use the parameter `ndig` for the number of digits of absolute accuracy. If the same methods are called without the `ndig` parameter, a default value of `ndig = 15` will be used.

```
public static double cdf (double x, double y, double rho, int ndig)
```

Computes the standard *binormal* distribution (16) with the method described in [2], where `ndig` is the number of decimal digits of accuracy provided (`ndig` ≤ 15). The code was translated from the Fortran program written by T. G. Donnelly and copyrighted by the ACM (see http://www.acm.org/pubs/copyright_policy/#Notice). The absolute error is expected to be smaller than 10^{-d} , where $d = \text{ndig}$.

```
public static double cdf (double mu1, double sigma1, double x,
                          double mu2, double sigma2, double y,
                          double rho, int ndig)
```

Computes the *binormal* distribution function (14) with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$, correlation $\rho = \text{rho}$ and `ndig` decimal digits of accuracy.

```
public static double barF (double mu1, double sigma1, double x,  
                           double mu2, double sigma2, double y,  
                           double rho, int ndig)
```

Computes the upper *binormal* distribution function (15) with parameters $\mu_1 = \text{mu1}$, $\mu_2 = \text{mu2}$, $\sigma_1 = \text{sigma1}$, $\sigma_2 = \text{sigma2}$, $\rho = \text{rho}$ and `ndig` decimal digits of accuracy.

```
public static double barF (double x, double y, double rho, int ndig)
```

Computes the upper *standard binormal* distribution function (15) with parameters $\rho = \text{rho}$ and `ndig` decimal digits of accuracy.

BiStudentDist

Extends the class `ContinuousDistribution2Dim` for the *standard bivariate Student's t* distribution [6, page 132]. The correlation between X and Y is ρ and the number of degrees of freedom is ν . Its probability density is

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \left[1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)} \right]^{-(\nu+2)/2}, \quad (17)$$

and the corresponding distribution function (the `cdf`) is

$$T_\nu(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^x dx \int_{-\infty}^y dy f(x, y). \quad (18)$$

We also define the upper distribution function called `barF` as

$$\bar{T}_\nu(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_x^\infty dx \int_y^\infty dy f(x, y). \quad (19)$$

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class BiStudentDist extends ContinuousDistribution2Dim
```

Constructor

```
public BiStudentDist (int nu, double rho)
```

Constructs a `BiStudentDist` object with correlation $\rho = \text{rho}$ and $\nu = \text{nu}$ degrees of freedom.

Methods

```
public static double density (int nu, double x, double y, double rho)
```

Computes the standard bivariate Student's t density function (17) with correlation $\rho = \text{rho}$ and $\nu = \text{nu}$ degrees of freedom.

```
public static double cdf (int nu, double x, double y, double rho)
```

Computes the standard bivariate Student's t distribution (18) using the method described in [4]. The code for the `cdf` was translated directly from the Matlab code written by Alan Genz and available from his web page at <http://www.math.wsu.edu/faculty/genz/homepage> (the code is copyrighted by Alan Genz and is included in this package with the kind permission of the author). The correlation is $\rho = \text{rho}$ and the number of degrees of freedom is $\nu = \text{nu}$.

```
public static double barF (int nu, double x, double y, double rho)
```

Computes the standard upper bivariate Student's t distribution (19).

```
public static double[] getMean (int nu, double rho)
```

Returns the mean vector $E[X] = (0, 0)$ of the bivariate Student's t distribution.

```
public static double[][] getCovariance (int nu, double rho)
```

Returns the covariance matrix of the bivariate Student's t distribution.

```
public static double[][] getCorrelation (int nu, double rho)
```

Returns the correlation matrix of the bivariate Student's t distribution.

```
protected void setParams (int nu, double rho)
```

Sets the parameters $\nu = \text{nu}$ and $\rho = \text{rho}$ of this object.

MultiNormalDist

Implements the abstract class `ContinuousDistributionMulti` for the *multinormal* distribution with mean vector μ and covariance matrix Σ . The probability density is

$$P[X = (x_1, \dots, x_d)] = \frac{1}{\sqrt{(2\pi)^d \det \Sigma}} \exp \left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \quad (20)$$

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class MultiNormalDist extends ContinuousDistributionMulti
```

Constructors

```
    public MultiNormalDist (double[] mu, double[][] sigma)
```

Methods

```
    public static double density (double[] mu, double[][] sigma, double[] x)
        Computes the density (20) of the multinormal distribution with parameters  $\mu = \text{mu}$  and  $\Sigma = \text{sigma}$ , evaluated at  $\mathbf{x}$ .
```

```
    public int getDimension()
        Returns the dimension  $d$  of the distribution.
```

```
    public static double[] getMean (double[] mu, double[][] sigma)
        Returns the mean  $E[X] = \mu$  of the multinormal distribution with parameters  $\mu$  and  $\Sigma$ .
```

```
    public static double[][] getCovariance (double[] mu, double[][] sigma)
        Computes the covariance matrix of the multinormal distribution with parameters  $\mu$  and  $\Sigma$ .
```

```
    public static double[][] getCorrelation (double[] mu, double[][] sigma)
        Computes the correlation matrix of the multinormal distribution with parameters  $\mu$  and  $\Sigma$ .
```

```
    public static double[] getMaximumLikelihoodEstimateMu (double[][] x,
                                                            int n, int d)
        Estimates the parameters  $\mu$  of the multinormal distribution using the maximum likelihood method based on the  $n$  observations of  $d$  components in table  $x[i][j]$ ,  $i = 0, 1, \dots, n-1$  and  $j = 0, 1, \dots, d-1$ .
```

```
    public static double[][] getMaximumLikelihoodEstimateSigma (double[][] x,
                                                                int n, int d)
        Estimates the parameters  $\Sigma$  of the multinormal distribution using the maximum likelihood method based on the  $n$  observations of  $d$  components in table  $x[i][j]$ ,  $i = 0, 1, \dots, n-1$  and  $j = 0, 1, \dots, d-1$ .
```

```
public double[] getMu()
```

Returns the parameter μ of this object.

```
public double getMu (int i)
```

Returns the i -th component of the parameter μ of this object.

```
public double[] [] getSigma()
```

Returns the parameter Σ of this object.

```
public void setParams (double[] mu, double[] [] sigma)
```

Sets the parameters μ and Σ of this object.

DirichletDist

Implements the abstract class `ContinuousDistributionMulti` for the *Dirichlet* distribution with parameters $(\alpha_1, \dots, \alpha_d)$, $\alpha_i > 0$. The probability density is

$$P[X = (x_1, \dots, x_d)] = \frac{\Gamma(\alpha_0) \prod_{i=1}^d x_i^{\alpha_i-1}}{\prod_{i=1}^d \Gamma(\alpha_i)} \quad (21)$$

where $x_i \geq 0$, $\sum_{i=1}^d x_i = 1$, $\alpha_0 = \sum_{i=1}^d \alpha_i$, and Γ is the Gamma function.

```
package umontreal.iro.lecuyer.probdistmulti;
```

```
public class DirichletDist extends ContinuousDistributionMulti
```

Constructors

```
    public DirichletDist (double[] alpha)
```

Methods

```
    public static double density (double[] alpha, double[] x)
```

Computes the density (21) of the Dirichlet distribution with parameters $(\alpha_1, \dots, \alpha_d)$.

```
    public static double[][] getCovariance (double[] alpha)
```

Computes the covariance matrix of the Dirichlet distribution with parameters $(\alpha_1, \dots, \alpha_d)$.

```
    public static double[][] getCorrelation (double[] alpha)
```

Computes the correlation matrix of the Dirichlet distribution with parameters $(\alpha_1, \dots, \alpha_d)$.

```
    public static double[] getMaximumLikelihoodEstimate (double[][] x,
                                                         int n, int d)
```

Estimates and returns the parameters $[\hat{\alpha}_1, \dots, \hat{\alpha}_d]$ of the Dirichlet distribution using the maximum likelihood method based on the n observations of d components in table $x[i][j]$, $i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, d-1$.

The equations of the maximum likelihood are defined in [1, Technical appendix]

$$L(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_k) = n \left(G(\alpha_0) - \sum_{i=1}^k G(\hat{\alpha}_i) \right) + \sum_{i=1}^k (\hat{\alpha}_i - 1) Z_i$$

where G is the logarithm of the gamma function and

$$\begin{aligned} \alpha_0 &= \sum_{i=1}^k \hat{\alpha}_i \\ Z_i &= \sum_{j=1}^n \ln(X_{i,j}) \quad \text{for } i = 1, \dots, k. \end{aligned}$$

```
public static double[] getMean (double[] alpha)
```

Computes the mean $E[X] = \alpha_i / \alpha_0$ of the Dirichlet distribution with parameters $(\alpha_1, \dots, \alpha_d)$, where $\alpha_0 = \sum_{i=1}^d \alpha_i$.

```
public double[] getAlpha()
```

Returns the parameters $(\alpha_1, \dots, \alpha_d)$ of this object.

```
public double getAlpha (int i)
```

Returns the i th component of the alpha vector.

```
public void setParams (double[] alpha)
```

Sets the parameters $(\alpha_1, \dots, \alpha_d)$ of this object.

References

- [1] A. N. Avramidis, A. Deslauriers, and P. L'Ecuyer. Modeling daily arrivals to a telephone call center. *Management Science*, 50(7):896–908, 2004.
- [2] T. G. Donnelly. Algorithm 462: Bivariate normal distribution. *Communications of the ACM*, 16(10):638, 1973.
- [3] Z. Drezner and G. O. Wesolowsky. On the computation of the bivariate normal integral. *Journal of Statistical Computation and Simulation*, 35:101–107, 1990.
- [4] A. Genz. Numerical computation of rectangular bivariate and trivariate normal and t probabilities. *Statistics and Computing*, 14:151–160, 2004. See <http://www.math.wsu.edu/faculty/genz/homepage>.
- [5] N. L. Johnson and S. Kotz. *Distributions in Statistics: Discrete Distributions*. Houghton Mifflin, Boston, 1969.
- [6] N. L. Johnson and S. Kotz. *Distributions in Statistics: Continuous Multivariate Distributions*. John Wiley, New York, NY, 1972.