Signals and Systems
Using MATLAB

Luis F. Chaparro

Chapter 10 - The Z-transform

What is in this chapter?

- Laplace transform of sampled signals and Z-transform
- Two- and one-sided Z-transforms
- Poles and zeros and region of convergence
- Convolution sum and transfer function
- Inverse Z-transform
- Solution of difference and differential equations

Laplace transform of sampled signals

Laplace transform of a sampled signal

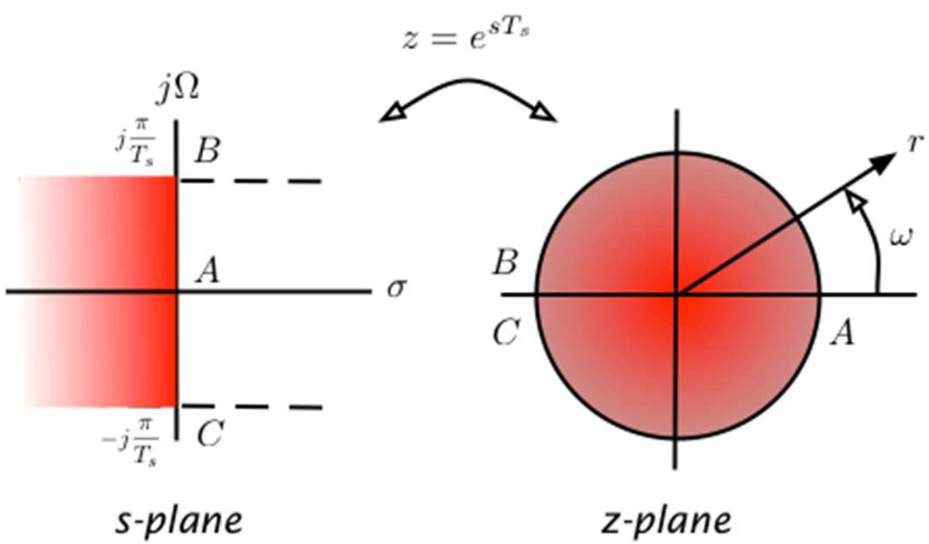
$$x(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

$$X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s)e^{-nsT_s}$$

Let $z = e^{sT_s}$,

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$$

the Z-transform of the sampled signal



Two-sided Z-transform $x[n]$, $-\infty < n < \infty$, *is*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad ROC$$

One-sided Z-transform *of* $x[n] = 0$ *for* $n < 0$, *or signals that are made causal by multiplying them with the unit-step signal* $u[n]$*:*

$$X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n} \qquad ROC_1$$

Two-sided Z-transform in terms of One-sided Z-transform:

$$X(z) = \mathcal{Z}\left(x[n]u[n]\right) + \mathcal{Z}\left(x[-n]u[n]\right)|_z - x[0]$$

$$ROC \text{ of } \; X(z): \quad \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$$

$$\mathcal{R}_1 \quad ROC \text{ (causal)} \quad \mathcal{Z}\left(x[n]u[n]\right)$$

$$\mathcal{R}_2 \quad ROC \text{ (anticausal)} \quad \mathcal{Z}\left(x[-n]u[n]\right)|_z$$

- Z-transform: transformation of sequence $\{x[n]\}$ into polynomial $X(z)$ so to each $x[n_0]$ we attach z^{-n_0}

$$\begin{array}{lllll} \cdots x[-2] & x[-1] & x[0] & x[1] & x[2]\cdots \\ \cdots x[-2]z^2 & x[-1]z^1 & x[0] & x[1]z^{-1} & x[2]z^{-2}\cdots \end{array}$$

$$x[n] = \sum_k x[k]\delta[n-k] \qquad \Leftrightarrow \qquad X(z) = \sum_k x[k]z^{-k}$$

- Two-sided Z-transform in terms of one-sides Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]u[n]z^{-n} + \sum_{n=-\infty}^{0} x[n]u[-n]z^{-n} - x[0]$$

$$= \mathcal{Z}\left(x[n]u[n]\right) + \underbrace{\sum_{m=0}^{\infty} x[-m]u[m]z^{m}}_{\mathcal{Z}(x[-n]u[n])|_z} - x[0]$$

<u>**Example**</u> Z-transform of $c[n] = \alpha^{|n|}, \quad 0 < \alpha < 1$

Causal $\quad \mathcal{Z}(c[n]u[n]) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$

ROC: $|\alpha z^{-1}| < 1$ or $|z| > \alpha$

Anti-causal $\quad \mathcal{Z}(c[-n]u[n])_z = \sum_{n=0}^{\infty} \alpha^n z^{n} = \frac{1}{1-\alpha z}$

ROC: $|\alpha z| < 1$ or $|z| < |1/\alpha|$

Two-sided Z-transform of $c[n]$:

$$C(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1 = \frac{z}{z-\alpha} - \frac{z}{(z-1/\alpha)} = \frac{(\alpha - 1/\alpha)z}{(z-\alpha)(z-1/\alpha)}$$

ROC: $|\alpha| < |z| < \left|\frac{1}{\alpha}\right|$

For $\alpha = 0.5$

$$C(z) = \frac{-1.5z}{(z-0.5)(z-2)} \qquad 0.5 < |z| < 2$$

Poles and Zeros

Poles of $X(z)$ are complex values $\{p_k\}$ such that

$$X(p_k) \to \infty$$

while the zeros of $X(z)$ are complex values $\{z_k\}$ that make

$$X(z_k) = 0$$

Example Find poles and zeros of

$$(i) \quad X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$
$$(ii) \quad X_2(z) = \frac{(z^{-1}-1)(z^{-1}+2)^2}{z^{-1}(z^{-2}+\sqrt{2}z^{-1}+1)}$$

Express $X_1(z)$ as function of positive powers of z:

$$\begin{aligned} X_1(z) &= \frac{z^3(1+2z^{-1}+3z^{-2}+4z^{-3})}{z^3} \\ &= \frac{z^3+2z^2+3z+4}{z^3} = \frac{N_1(z)}{D_1(z)} \end{aligned}$$

poles roots of $D_1(z) = z^3 = 0$ or $z = 0$, triple
zeros are the roots of $N_1(z) = z^3 + 2z^2 + 3z + 4 = 0$

Expressing $X_2(z)$ as function of positive powers of z,

$$\begin{aligned} X_2(z) &= \frac{z^3(z^{-1}-1)(z^{-1}+2)^2}{z^3(z^{-1}(z^{-2}+\sqrt{2}z^{-1}+1))} \\ &= \frac{(1-z)(1+2z)^2}{1+\sqrt{2}z+z^2} = \frac{N_2(z)}{D_2(z)} \end{aligned}$$

poles of $X_2(z)$ are roots of $D_2(z) = 1 + \sqrt{2}z + z^2 = 0$
zeros of $X_2(z)$ are the roots of $N_2(z) = (1-z)(1+2z)^2 = 0$

The ROC of the z-transform of a

1. **finite support signal** $x[n]$ *is the whole z-plane, excluding the origin* $z = 0$ *and/or* $z = \pm\infty$ *(depending on the boundaries of the support)*
2. **causal signal** $x[n]$ *is* $|z| > R_1$ *(outside of a circle of radius* R_1*),* R_1 *being the largest radius of the poles of* $X(z)$
3. **anti-causal signal** $x[n]$ *is* $|z| < R_2$ *(inside of a circle of radius* R_2*),* R_2 *being the smallest radius of the poles of* $X(z)$*,*
4. **non-causal signal** $x[n]$ *is* $R_1 < |z| < R_2$*, or the inside of a torus of inside radius* R_1 *and outside radius* R_2 *corresponding to the maximum and minimum radii of the poles of* $X_c(z)$ *and* $X_a(z)$*, Z-transforms of the causal and anticausal components of* $x[n]$*.*

Example Z-transform of

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} X(z) &= \sum_{n=0}^{9} 1\ z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}} = \frac{z^{10} - 1}{z^9(z-1)} \\ &= \frac{\prod_{k=1}^{9}(z - e^{j\pi k/5})}{z^9} \end{aligned}$$

last expression is due to a pole/zero cancellation
zeros are the roots of

$$z^{10} - 1 = 0 \quad \Rightarrow z_k = e^{j2\pi k/10},\ k = 0 \cdots 9$$

zero $z_0 = 1$, cancels the pole at 1
ROC: whole z-plane except $z = 0$

Example 5. Find ROCs of the Z-transforms of the following signals

$$(i) \quad x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$(ii) \quad x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

$x_1[n]$ causal

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$$

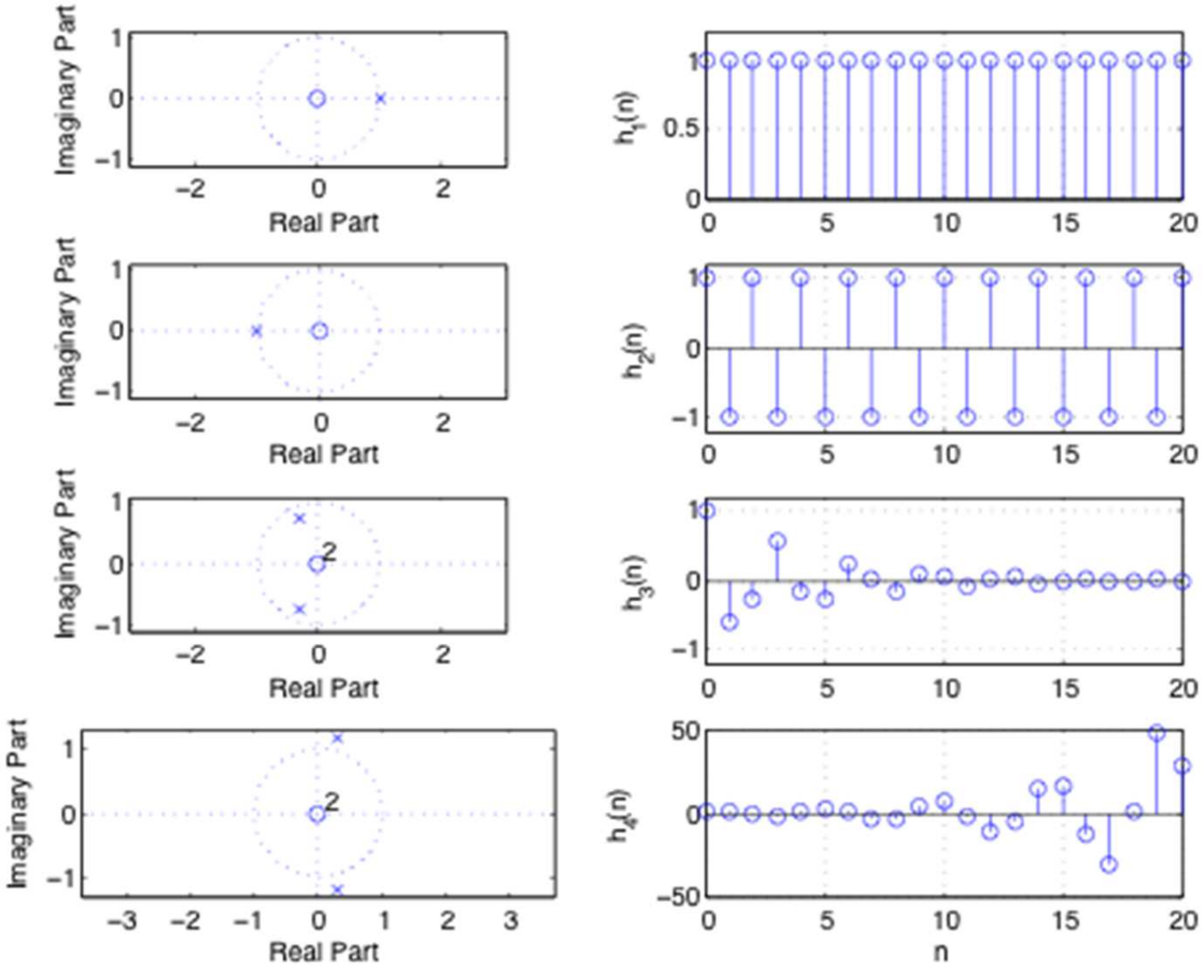
provided $|0.5z^{-1}| < 1$ or $\mathcal{R}_1 : |z| > 0.5$ outside of circle of radius 0.5

$x_2[n]$ anti-causal

$$X_2(z) = -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} = -\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{-m} z^m + 1 = -\sum_{m=0}^{\infty} 2^m z^m + 1 = \frac{-1}{1-2z} + 1 = \frac{z}{z-0.5}$$

ROC: $\mathcal{R}_2 : |z| < 0.5$

$X_1(z) = X_2(z)$ although $x_1[n] \neq x_2[n]$, ROCs differentiate them
Z-transform of $x_1[n] + x_2[n]$ does not exist given that the intersection of $\mathcal{R}_1$ and $\mathcal{R}_2$ is empty



Effect of pole location on the inverse Z-transform: (from top to bottom) if pole is at $z = 1$ the signal is $u(n)$, constant for $n \geq 0$; if pole is at $z = -1$ the signal is a cosine of frequency π continuously changing, constant amplitude; when poles are complex, if inside the unit circle the signal is a decaying modulated exponential, and if outside the unit circle the signal is a growing modulated exponential.

- *The Z-transform is a* **linear transformation**

$$\mathcal{Z}(ax[n] + by[n]) = a\mathcal{Z}(x[n]) + b\mathcal{Z}(y[n])$$

- *Exponentials —For real* $\alpha = |\alpha|e^{j\omega_0}$, *for* $\omega_0 = 0, \pi$

$$x[n] = \alpha^n u[n] \quad \Leftrightarrow \quad X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \quad \text{ROC: } |z| > |\alpha|$$

  - *when* $\alpha > 0$ *then* $\omega_0 = 0$ *and the signal is less and less damped as* $\alpha \to \infty$, *and*
  - *when* $\alpha < 0$ *then* $\omega_0 = \pi$ *and the signal is a modulated exponential that grows as* $\alpha \to -\infty$

- *Sinusoids*

$$\cos(\omega_0 n)u[n] \quad \Leftrightarrow \quad \frac{z(z-\cos(\omega_0))}{(z-e^{j\omega_0})(z-e^{-j\omega_0})} \quad ROC : |z| > 1$$

$$\sin(\omega_0 n)u[n] \quad \Leftrightarrow \quad \frac{z\sin(\omega_0)}{(z-e^{j\omega_0})(z-e^{-j\omega_0})} \quad ROC : |z| > 1$$

- *The Z-transform pair*

$$r^n \cos(\omega_0 n + \theta)u[n] \quad \Leftrightarrow \quad \frac{z(z\cos(\theta) - r\cos(\omega_0 - \theta))}{(z-re^{j\omega_0})(z-re^{-j\omega_0})}$$

damping indicated by r *and frequency given by* ω_0

- *Double poles*

$$nx[n]u[n] \quad \Leftrightarrow \quad -z\frac{dX(z)}{dz}.$$

Convolution Sum and Transfer Function

The output $y[n]$ of a causal LTI system is computed using the convolution sum

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

where $x[n]$ is a causal input and $h[n]$ the impulse response of the system. The Z-transform of $y[n]$ is the product

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

and the transfer function of the system is thus defined as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{ output } y[n]]}{\mathcal{Z}[\text{ input } x[n]]}$$

i.e., $H(z)$ transfers the input $X(z)$ into the output $Y(z)$.

Remarks

- Whenever multiplying two polynomials $X_1(z)$ and $X_2(z)$, of finite or infinite order, the coefficients of the resulting polynomial can be obtained by means of the convolution sum. For instance,

$$X_1(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$$
$$X_2(z) = 1 + b_1 z^{-1}$$

$$Y(z) = X_1(z)X_2(z) = 1 + (b_1 + a_1)z^{-1} + (a_1 b_1 + a_2)z^{-2} + a_2 b_1 z^{-3}$$

Convolution sum of $[1 \;\; a_1 \;\; a_2]$ and $[1 \;\; b_1]$, from coefficients of $X_1(z)$ and $X_2(z)$, is $[1 \;\; (a_1+b_1) \;\; (a_2+b_1 a_1) \;\; a_2]$, also order $Y(z)$=order $X_1(z)$ +order of $X_2(z)$ -1

- FIR filters implemented by convolution sum:

N^{th} – order FIR

$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k] \qquad x[n] \text{ input, } y[n] \text{ output}$$

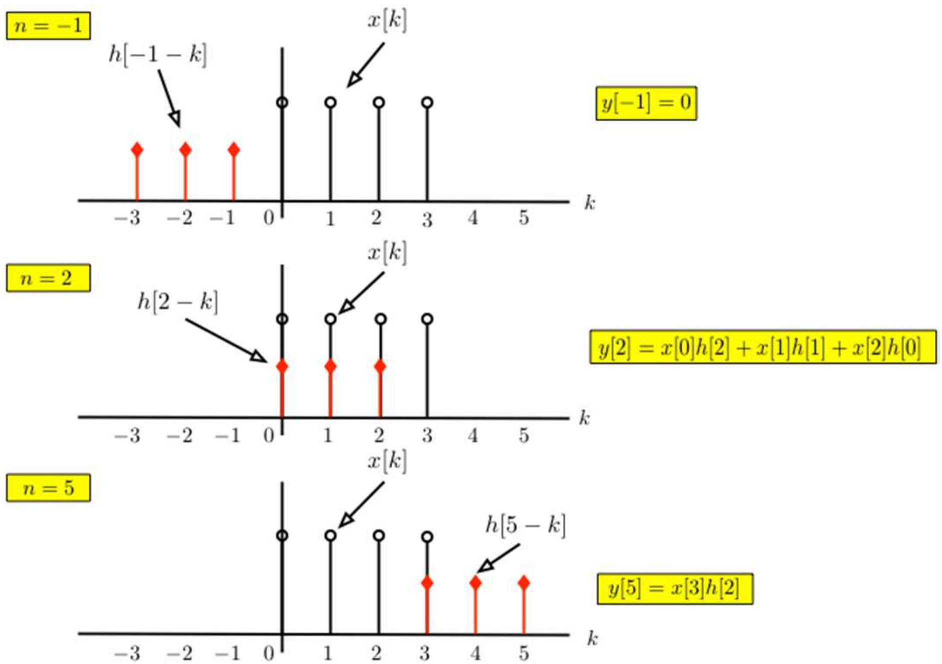
$$\text{Impulse resp. } h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k] \quad \Rightarrow \quad h[n] = b_n,\ n = 0, \cdots, N-1$$

$$\text{so that } y[n] = \sum_{k=0}^{N-1} h[k] x[n-k]$$

Example Find output of FIR filter

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1] + x[n-2]\right)$$

for an input $x[n] = u[n] - u[n-4]$



Convolution sum property:

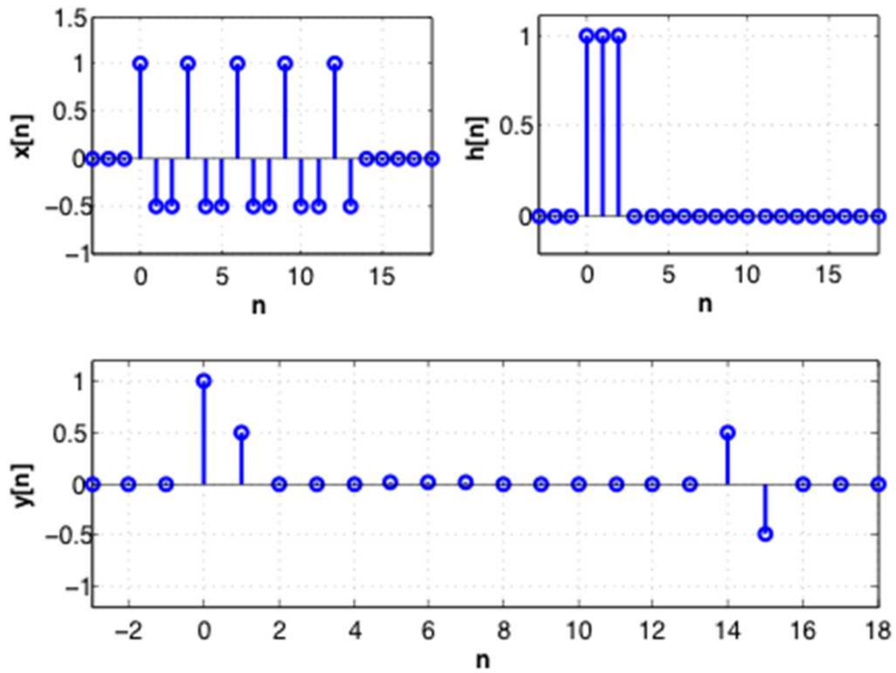
$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$
$$H(z) = \frac{1}{2}[1 + z^{-1} + z^{-2}]$$
$$Y(z) = X(z)H(z) = \frac{1}{2}(1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5})$$

so $y[0] = 0.5$, $y[1] = 1$, $y[2] = 1.5$, $y[3] = 1.5$, $y[4] = 1$, and $y[5] = 0.5$

Example FIR filter with impulse response

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Find filter output $y[n]$ for input $x[n] = \cos(2\pi n/3)(u[n] - u[n-14])$



Convolution property approach:

$$Y(z) = X(z)H(z) = X(z)(1 + z^{-1} + z^{-2}) = X(z) + X(z)z^{-1} + X(z)z^{-2}.$$

z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}	z^{-5}	z^{-6}	z^{-7}	z^{-8}	z^{-9}	z^{-10}	z^{-11}	z^{-12}	z^{-13}	z^{-14}	z^{-15}
1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5		
	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	
		1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5	−0.5	1	−0.5

Adding vertically these coefficients we obtain

$$Y(z) = 1 + 0.5z^{-1} + 0z^{-2} + \cdots + 0z^{-13} + 0.5z^{-14} - 0.5z^{-15} = 1 + 0.5z^{-1} + 0.5z^{-14} - 0.5z^{-15}$$

Example First-order IIR system

$$y[n] = 0.5y[n-1] + x[n] \qquad \text{input } x[n], \text{ output } y[n]$$

Determine transfer function, impulse and the unit-step responses, if system is BIBO stable and transient and steady state responses of the system if possible

Transfer function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 - 0.5z^{-1}}$

impulse response $h[n] = \mathcal{Z}^{-1}[H(z)] = 0.5^n u[n]$

unit-step response $x[n] = u[n]$

$$Y(z) = H(z)X(z) = \frac{1}{(1 - 0.5z^{-1})(1 - z^{-1})} = \frac{-1}{1 - 0.5z^{-1}} + \frac{2}{1 - z^{-1}}$$

$$y[n] = -0.5^n u[n] + 2u[n]$$

BIBO stable

$$\sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2, \quad h[n] \text{ absolutely summable, also}$$

$$H(z) = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}, \quad \text{pole } z = 0.5 \text{ inside unit circle}$$

$$\text{steady state}: \quad n \to \infty, \quad y[n] = 2$$

$$\text{transient}: \quad -0.5^n u[n]$$

Example FIR system

$$y[n] = \frac{1}{3}[x[n] + x[n-1] + x[n-2]] \qquad x[n] \text{ input } \; y[n] \text{ output}$$

Find transfer function, impulse response and determine if system is BIBO stable or not

Transfer function

$$H(z) = \frac{1}{3}[1 + z^{-1} + z^{-2}] = \frac{z^2 + z + 1}{3z^2}$$

impulse response

$$h[n] = \frac{1}{3}[\delta[n] + \delta[n-1] + \delta[n-2]]$$

$h[n]$ absolutely summable so system is BIBO

Non-recursive or FIR Systems. *The impulse response $h[n]$ of an FIR or non-recursive system*

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

has finite length and is given by

$$h[n] = b_0 \delta[n] + b_1 \delta[n-1] + \cdots + b_M \delta[n-M]$$

Its transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^M}$$

with all its poles at the origin $z = 0$ (multiplicity M) and as such the system is BIBO stable.

Recursive or IIR Systems. The impulse response $h[n]$ of an IIR or recursive system

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

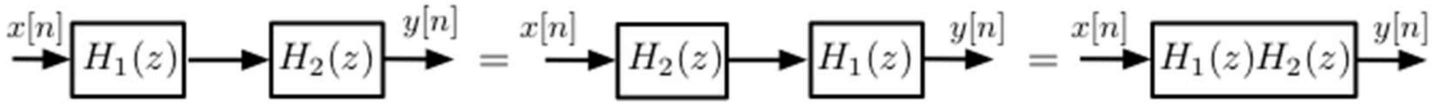
has (possible) infinite length and is given by

$$\begin{aligned} h[n] &= \mathcal{Z}^{-1}[H(z)] = \mathcal{Z}^{-1}\left[\frac{\sum_{m=0}^{M} b_m z^{-m}}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right] \\ &= \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}\right] = \sum_{\ell=0}^{\infty} h[\ell]\delta[n-\ell] \end{aligned}$$

where $H(z)$ is the transfer function of the system. If the poles of $H(z)$ are inside the unit circle, or $A(z) \neq 0$ for $|z| \geq 1$, the system is BIBO stable.

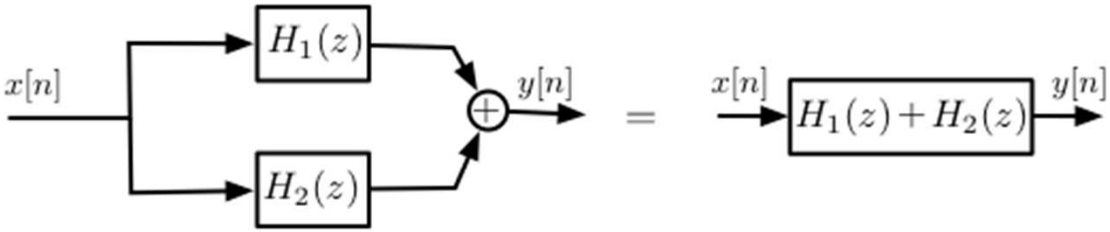
Interconnection of discrete-time systems

Cascade



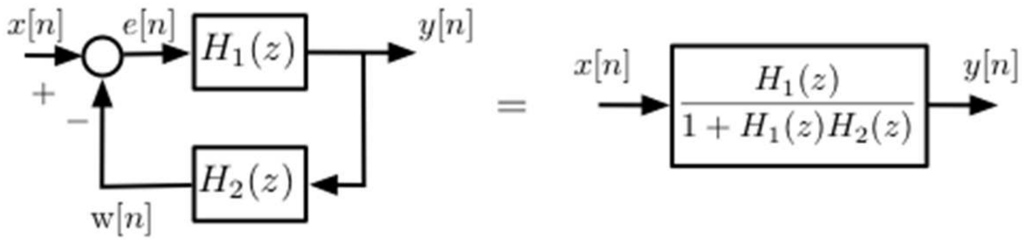
(a)

Parallel



(b)

Negative feedback



(c)

Negative feedback connection:

feed-forward path $Y(z) = H_1(z)E(z)$

error function $E(z) = X(z) - Y(z)$

feedback path $W(z) = \mathcal{Z}[w[n]] = H_2(z)Y(z)$

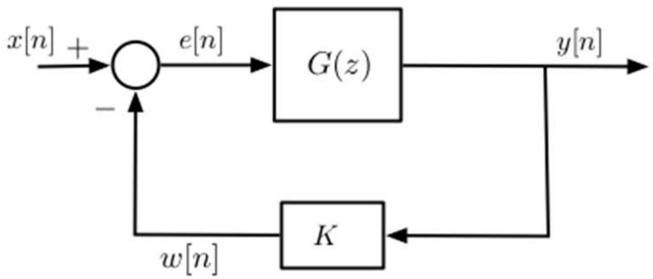
$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Initial and Final Value Properties

If $X(z)$ is the Z-transform of a causal signal $x[n]$ then

$$\text{Initial value:} \quad x[0] = \lim_{z\to\infty} X(z)$$
$$\text{Final value:} \quad \lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z) \tag{1}$$

Example Plant $G(z) = 1/(1-0.5z^{-1})$ connected with constant feedback gain K. Reference signal $x[n] = u[n]$, determine behavior of error signal $e[n]$. Effect of the feedback on an unstable plant $G(z) = 1/(1-z^{-1})$?



Error signal $E(z) = X(z) - W(z) = X(z) - KG(z)E(z)$

$$X(z) = 1/(1-z^{-1}) \quad \Rightarrow \quad E(z) = \frac{X(z)}{1+KG(z)} = \frac{1}{(1-z^{-1})(1+KG(z))}$$

initial value of the error $e[0] = \lim_{z\to\infty} E(z) = \dfrac{1}{1+K}$ since $G(\infty) = 1$

Steady–state or final error

$$\lim_{n\to\infty} e[n] = \lim_{z\to 1} \frac{(z-1)X(z)}{1+KG(z)} = \frac{1}{1+2K} \quad \text{since } G(1) = 2$$

K large, $e[0] = e[\infty] = 0$

For unstable plant $G(z) = 1/(1-z^{-1})$

$e[0] = 1/(1+K)$, steady-state error $e[\infty] \to 0$ since $G(1) \to \infty$

One-sided Z-transform Inverse

Inspection

$$X(z) = 1 + 2z^{-10} + 3z^{-20} \quad \Rightarrow \quad \text{inverse}: \quad x[n] = \delta[n] + 2\delta[n-10] + 3\delta[n-20]$$

Long-division

Rational function $X(z) = B(z)/A(z)$, having as ROC the outside of a circle of radius R (i.e., $x[n]$ is causal):

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

then the inverse is the sequence $\{x[0], x[1], x[2], \cdots\}$ or

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Example Inverse Z-transform of

$$X(z) = \frac{1}{1+2z^{-2}} \qquad |z| > \sqrt{2}$$

Let $X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$

$$1 = (1+2z^{-2})\underbrace{(x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots)}_{X(z)}$$

$$1 = \left(x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots\right) + \left(2x[0]z^{-2} + 2x[1]z^{-3} + \cdots\right)$$

$$\begin{aligned}
&x[0] = 1; \quad x[1] = 0 \\
&x[2] + 2x[0] = 0 \quad \Rightarrow \quad x[2] = -2; \quad x[3] + 2x[1] = 0 \quad \Rightarrow \quad x[3] = 0 \\
&x[4] + 2x[2] = 0 \quad \Rightarrow \quad x[4] = (-2)^2 \\
&\qquad \vdots
\end{aligned}$$

Partial Fraction Expansion

- PFE same as for Laplace
- Given $X(z) = N(z)/D(z)$ must be proper rational (i.e., $N(z)$ of lower order than $D(z)$), if not use long division

Example

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

Obtain an expansion of $X(z)$ containing a proper rational term

$$\text{Long division} \quad X(z) = 1 + \underbrace{\frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}}_{\text{proper rational}}$$

Inverse Z-transform

$$x[n] = \delta[n] + \mathcal{Z}^{-1}\left[\frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}\right]$$

Example Find inverse Z-transform of

$$X(z) = \frac{1+z^{-1}}{(1+0.5z^{-1})(1-0.5z^{-1})} = \frac{z(z+1)}{(z+0.5)(z-0.5)} \qquad |z| > 0.5$$

using negative and positive powers of z expressions

- Negative powers

$$\begin{aligned} X(z) &= \frac{1+z^{-1}}{(1+0.5z^{-1})(1-0.5z^{-1})} \\ &= \frac{A}{1+0.5z^{-1}} + \frac{B}{1-0.5z^{-1}} \\ A &= X(z)(1+0.5z^{-1})|_{z^{-1}=-2} = -0.5 \\ B &= X(z)(1-0.5z^{-1})|_{z^{-1}=2} = 1.5 \end{aligned}$$

$$x[n] = [-0.5(-0.5)^n + 1.5(0.5)^n]u[n]$$

- Positive powers – $X(z)$ not proper, let

$$\begin{aligned} \frac{X(z)}{z} &= \frac{z+1}{(z+0.5)(z-0.5)} \\ &= \frac{C}{z+0.5} + \frac{D}{z-0.5} \\ C &= \frac{X(z)}{z}(z+0.5)|_{z=-0.5} = -0.5 \\ D &= \frac{X(z)}{z}(z-0.5)|_{z=0.5} = 1.5 \\ X(z) &= \frac{-0.5z}{z+0.5} + \frac{1.5z}{z-0.5} \\ x[n] &= [-0.5(-0.5)^n + 1.5(0.5)^n]u[n] \end{aligned}$$

Checks:

$$x[0] = 1 = \lim_{z \to \infty} X(z)$$

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1}(z-1)X(z) = 0$$

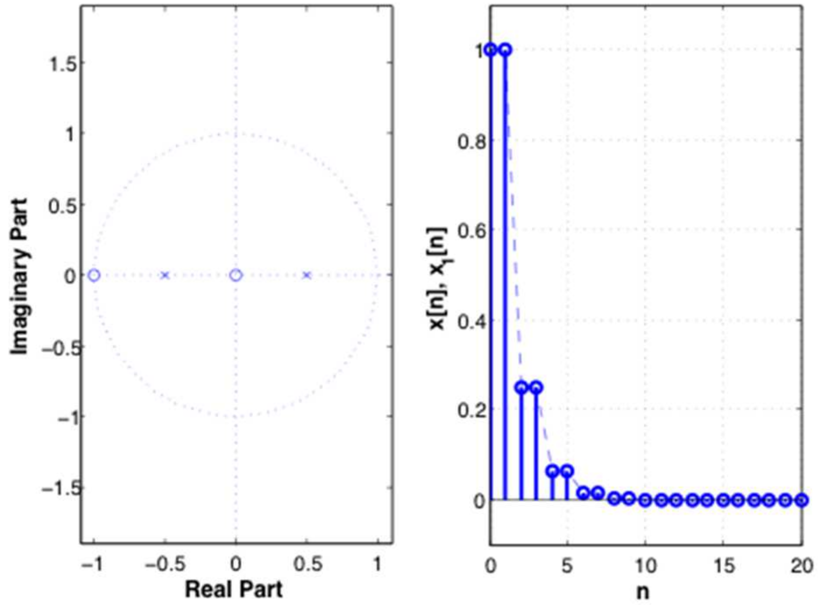
Partial Fraction Expansion with MATLAB

$$\begin{aligned} X(z) &= \frac{z(z+1)}{(z-0.5)(z+0.5)} = \frac{(1+z^{-1})}{(1-0.5z^{-1})(1+0.5z^{-1})} \qquad |z| > 0.5 \\ &= \frac{1.5}{1-0.5z^{-1}} - \frac{0.5}{1+0.5z^{-1}} \end{aligned}$$

```
% Two methods for inverse Z-transform
%
p1=poly(0.5); p2=poly(-0.5); % generation of terms in denominator
a=conv(p1,p2) % denominator coefficients
z1=poly(0); z2=poly(-1); % generation of terms in numerator
b=conv(z1,z2) % numerator coefficients
z=roots(b) % zeros of X(z)
[r,p,k]=residuez(b,a) % partial fraction expansion, poles and gain
zplane(b,a) % plot of poles and zeros
d=[1 zeros(1,99)]; % impulse delta[n]
x=filter(b,a,d); % x[n] computation from filter

n=0:99;
x1=r(1)*p(1).^n+r(2)*p(2).^n; % x[n] computation from residues
```

```
a = 1      0    -0.25
b = 1      1     0
z = 0
   -1
r = 1.5000
   -0.5000
p = 0.5000
   -0.5000
```



<u>Example</u>

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > a$$
$$x[n] = na^n u[n]$$

PFE a la MATLAB

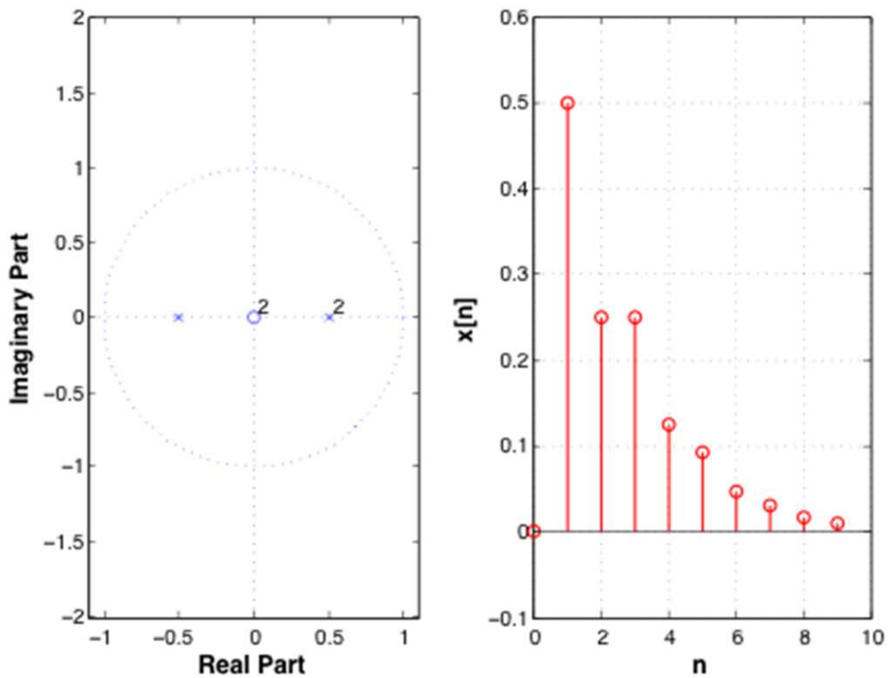
$$X(z) = \frac{r_1}{1-az^{-1}} + \underbrace{\frac{r_2}{(1-az^{-1})^2}}_{\text{not in table of Z-trans}} \quad , \qquad r_1 = -1, r_2 = 1$$

Our method

$$X(z) = \frac{A}{1-az^{-1}} + \frac{Bz^{-1}}{(1-az^{-1})^2}$$
$$A = r_1 + r_2, \quad B - Aa = -r_1 a \text{ or } B = ar_2$$
$$x[n] = [(r_1+r_2)a^n + nr_2a^n]u[n] = na^n u[n]$$

$$X(z) = \frac{0.5z^{-1}}{1 - 0.5z^{-1} - 0.25z^{-2} + 0.125z^{-3}}$$

```
%  Inverse Z-transform --- multiple poles
%
b=[0 0.5 0 0 ]; a=[1 -0.5 -0.25 0.125]
[r,p,k]=residuez(b,a) % partial fraction expansion, poles and gain
zplane(b,a) % plot of poles and zeros
n=0:99; xx=p(1).^n; yy=xx.*n;
x1=(r(1)+r(2)).*xx+r(2).*yy+r(3)*p(3).^n; % inverse computation
```



Solution of Difference Equations

If $x[n]$ has a one-sided Z-transform $X(z)$, then $x[n-N]$ has the following one-sided Z-transform

$$\mathcal{Z}[x[n-N]] = z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N]$$

Example System represented by second-order difference equation with constant coefficients

$$y[n] - a_1 y[n-1] - a_2 y[n-2] = x[n] + b_1 x[n-1] + b_2 x[n-2] \qquad n \geq 0$$

input $x[n]$, output $y[n]$

initial conditions $y[-1],\ y[-2]$

$$\mathcal{Z}(y[n] - a_1 y[n-1] - a_2 y[n-2]) = \mathcal{Z}(x[n] + b_1 x[n-1] + b_2 x[n-2])$$
$$Y(z) - a_1(z^{-1}Y(z) + y[-1]) - a_2(z^{-2}Y(z) + y[-1]z^{-1} + y[-2]) = X(z)(1 + b_1 z^{-1} + b_2 z^{-2})$$
$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2}) = (y[-1](a_1 + a_2 z^{-1}) + a_2 y[-2]) + X(z)(1 + b_1 z^{-1} + b_2 z^{-2})$$

solving for $Y(z)$ we have

$$Y(z) = \underbrace{\frac{X(z)(1 + b_1 z^{-1} + b_2 z^{-2})}{1 - a_1 z^{-1} - a_2 z^{-2}}}_{\text{zero-state response}} + \underbrace{\frac{y[-1](a_1 + a_2 z^{-1}) + a_2 y[-2])}{1 - a_1 z^{-1} - a_2 z^{-2}}}_{\text{zero-input response}}$$

Example Complete response of

$$\begin{aligned}
&y[n] + y[n-1] - 4y[n-2] - 4y[n-3] = 3x[n] \qquad n \geq 0 \\
&y[-1] = 1 \\
&y[-2] = y[-3] = 0 \\
&x[n] = u[n]
\end{aligned}$$

Is corresponding discrete-time system BIBO stable? steady state response?

$$Y(z)[1 + z^{-1} - 4z^{-2} - 4z^{-3}] = 3X(z) + [-1 + 4z^{-1} + 4z^{-2}]$$

Let $A(z) = 1 + z^{-1} - 4z^{-2} - 4z^{-3} = (1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})$

$$Y(z) = \underbrace{3\frac{X(z)}{A(z)}}_{\text{zero-state resp.}} + \underbrace{\frac{-1 + 4z^{-1} + 4z^{-2}}{A(z)}}_{\text{zero-input resp.}} \qquad |z| > 2$$

Transfer funct. $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{3}{A(z)}$

poles $z = -1,\ z = -2,\ z = 2 \quad \Rightarrow \quad h[n] = \mathcal{Z}^{-1}[H(z)]$ not absolutely summable

System is not BIBO stable

$$\begin{aligned}
Y(z) &= \frac{2 + 5z^{-1} - 4z^{-3}}{(1 - z^{-1})(1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})} \\
&= \frac{-0.5}{1 - z^{-1}} + \frac{-1/6}{1 + z^{-1}} + \frac{0}{1 + 2z^{-1}} + \frac{8/3}{1 - 2z^{-1}} \\
y[n] &= \left(-0.5 - \frac{1}{6}(-1)^n + \frac{8}{3}2^n\right) u[n]
\end{aligned}$$

there is no steady-state response.

Example

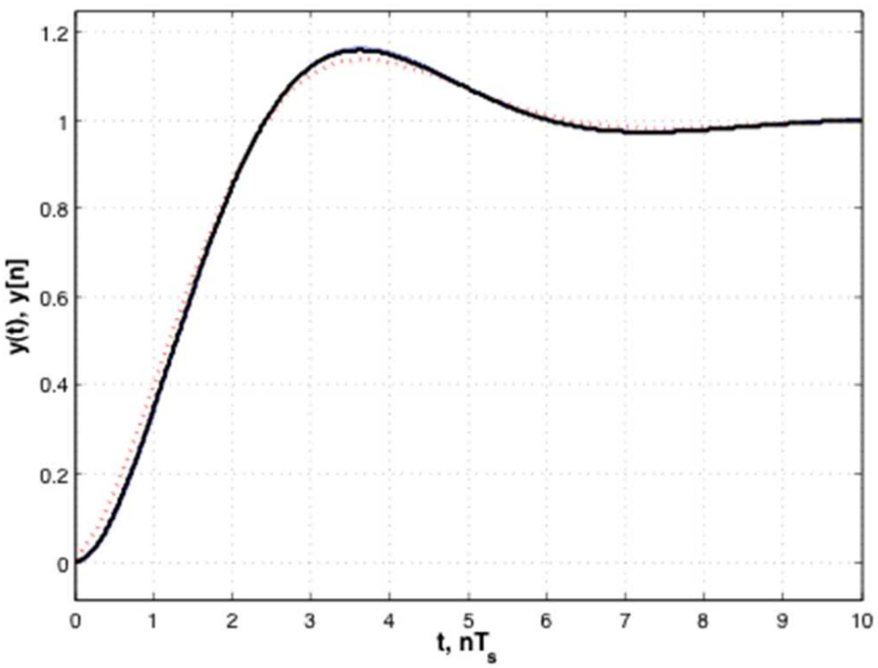
$$\text{RLC circuit}: \quad \frac{d^2v_c(t)}{dt^2} + \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$
$$v_c(t) \text{ output}, \; v_s(t) = u(t), \text{ input, ICs} = 0$$

$$V_c(s) = \frac{V_s(s)}{1+s+s^2} = \frac{1}{s(s^2+s+1)} = \frac{1}{s((s+0.5)^2+3/4)}$$
$$v_c(t) = [A + Be^{-0.5t}\cos(\sqrt{3}/2t+\theta)]u(t)$$

$$\text{Let} \quad \frac{dv_c(t)}{dt} \approx \frac{v_c(t) - v_c(t-T_s)}{T_s}$$
$$\frac{d^2v_c(t)}{dt^2} = \frac{d\frac{dv_c(t)}{dt}}{dt} \approx \frac{d(v_c(t)-v_c(t-T_s))/T_s}{dt} \approx \frac{v_c(t) - 2v_c(t-T_s) + v_c(t-2T_s)}{T_s^2}$$
$$\text{difference equation} \quad \left(\frac{1}{T_s^2} + \frac{1}{T_s} + 1\right) v_c(nT_s) - \left(\frac{2}{T_s^2} + \frac{1}{T_s}\right) v_c((n-1)T_s) + \left(\frac{1}{T_s^2}\right) v_c((n-2)T_s) = v_s(nT$$

$$T_s = 1, \quad 3v_c[n] - 3v_c[n-1] + v_c[n-2] = v_s[n] \qquad n \geq 0, \quad \text{ICs: zero}$$

$$[3 - 3z^{-1} + z^{-2}]V_c(z) = \frac{1}{1-z^{-1}}$$
$$V_c(z) = \frac{z^3}{(z-1)(3z^2-3z+1)}$$




```
% Solution of differential equation
syms s
vc=ilaplace(1/(s^3+s^2+s));  % exact solution
ezplot(vc,[0,10]);grid; hold on % plotting of exact solution
Ts=0.1; % sampling period
a1=1/Ts^2+1/Ts+1;a2=-2/Ts^2-1/Ts;a3=1/Ts^2; % coefficients
a=[1 a2/a1 a3/a1];b=1;
t=0:Ts:10; N=length(t);
vs=ones(1,N); % input
vca=filter(b,a,vs);vca=vca/vca(N); % solution
```

Inverse of Two-sided Z-transforms

- Relate poles to causal and anti-causal components
- The region of convergence plays a very important role in making this determination
- Inverse is found by looking for the causal and the anti-causal partial fraction expansion components in a Z-transform table

Example Inverse Z-transform of

$$X(z) = \frac{2z^{-1}}{(1-z^{-1})(1-2z^{-1})^2} \qquad 1 < |z| < 2$$

$$\begin{aligned}
&X(z) \;\; \text{zeros } z_{1,2} = 0, \text{ poles } z = 1, \; z = 2 \text{ (double)} \\
&\text{associate with pole } z = 1, \; ROC_1 = \mathcal{R}_1 : |z| > 1 \;\; \text{(causal signal)} \\
&\text{associate with pole } z = 2, \; ROC_2 = \mathcal{R}_2 : |z| < 2 \;\text{(anti-causal signal)}
\end{aligned}$$

,

$$X(z) = \underbrace{\frac{A}{1-z^{-1}}}_{\mathcal{R}_1:|z|>1} + \underbrace{\left[\frac{B}{1-2z^{-1}} + \frac{Cz^{-1}}{(1-2z^{-1})^2}\right]}_{\mathcal{R}_2:|z|<2}$$

$$1 < |z| < 2 = \mathcal{R}_1 \cap \mathcal{R}_2$$

$$\begin{aligned}
&A = X(z)(1-z^{-1})|_{z^{-1}=1} = 2 \\
&C = X(z)\frac{(1-2z^{-1})^2}{z^{-1}}|_{z^{-1}=0.5} = 4 \\
&\text{for } z^{-1} = 0, X(0) = A + B = 0 \quad \Rightarrow \quad B = -A = -2 \\
&x[n] = \underbrace{2u[n]}_{\text{causal}} + \underbrace{\left[-2^{(n+1)}u[-n-1] + 2^{(n+2)}nu[-n-1]\right]}_{\text{anti-causal}}
\end{aligned}$$

What have we accomplished?

- Connection of Laplace and Z-transforms
  - Significance of poles and zeros in z-plane and poles for ROC
- Solution of difference and differential equations with initial conditions
- Transfer function of recursive and non-recursive filters

Where do we go from here?

- Fourier analysis of discrete-time systems
  - Frequency representation of discrete-time systems
- Application of Z-transform to control