Signals and Systems Using MATLAB

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Chapter 9 - Discrete-time Signals and Systems

What is in this chapter?

- Discrete-time signals
  - Periodicity, symmetry, energy and power
  - Basic discrete-time signals
- Discrete-time LTI systems
  - Recursive and non-recursive systems
  - Difference representation of systems
- Convolution sum
- Causality and stability

A **discrete-time signal** $x[n]$ *can be thought of as a real- or complex-valued function of the integer sample index n:*

$$x[.] : \mathcal{I} \to \mathcal{R} \quad (\mathcal{C})$$
$$n \qquad x[n]$$

- $x(nT_s)$ depends on n once T_s is known and is only defined at nT_s
- Most discrete-time signals are obtained by sampling continuous-time signals, but there are inherently discrete signals

Examples

(a) Sampling $x(t) = 3\cos(2\pi t + \pi/4) \qquad -\infty < t < \infty$ using T_s satisfying Nyquist sampling rate condition

$$T_s \leq \frac{\pi}{\Omega_{max}} = \frac{\pi}{2\pi} = 0.5$$

for largest allowed sampling period $T_s = 0.5$

$$x[n] = 3\cos(2\pi t + \pi/4)|_{t=0.5n} = 3\cos(\pi n + \pi/4) \qquad -\infty < n < \infty$$

(b) Fibonacci sequence $\{x[n]\}$

$$
\begin{aligned}
x[n] &= x[n-1] + x[n-2] \qquad n \geq 2 \\
x[0] &= 0 \\
x[1] &= 1
\end{aligned}
$$

it has been used to model different biological systems

$$
\begin{aligned}
x[2] &= 1 + 0 = 1 \\
x[3] &= 1 + 1 = 2 \\
x[4] &= 2 + 1 = 3 \\
x[5] &= 3 + 2 = 5 \\
&\vdots \\
x[n] &= x[n-1] + x[n-2]
\end{aligned}
$$

Sequence is purely discrete as it is not related to a continuous–time signal

Periodic and Aperiodic Signals

$x[n]$ is periodic if

- *it is defined for all possible values of n, $-\infty < n < \infty$, and*
- *there is a positive integer N, the period of $x[n]$, such that*

$$x[n + kN] = x[n]$$

for any integer k.

Aperiodic signals are non-periodic

Sum $z[n] = x[n] + y[n]$ of periodic signals $x[n]$ of period N_1, and $y[n]$ of period N_2 is periodic if

$$\frac{N_2}{N_1} = \frac{p}{q} \quad \textit{is rational}$$

i.e., p and q are integers, and not divisible by each other
The period of $z[n]$ is $qN_2 = pN_1$

Example $z[n] = v[n]+w[n]+y[n]$, $v[n]$, $w[n]$ and $y[n]$ periodic of periods $N_1 = 2$, $N_2 = 3$ and $N_3 = 4$, respectively. Determine if $z[n]$ is periodic, and if so its period

$x[n] = v[n] + w[n]$, so that $z[n] = x[n] + y[n]$

$x[n]$ is periodic since $N_2/N_1 = 3/2$ is a rational number, and its period is $N_4 = 3N_1 = 2N_2 = 6$

$z[n]$ is also periodic since

$$\frac{N_4}{N_3} = \frac{6}{4} = \frac{3}{2}$$

is rational. Its period is $N = 2N_4 = 3N_3 = 12$, i.e.,

$$z[n+12] = v[n+6N_1]+w[n+4N_2]+y[n+3N_3] = v[n]+w[n]+y[n] = z[n] \qquad \square$$

Discrete-time Periodic Sinusoids

Periodic discrete-time sinusoids, of period N, are of the form

$$x[n] = A\cos\left(\frac{2\pi m}{N}n + \theta\right) \qquad -\infty < n < \infty$$

where the discrete frequency is $\omega_0 = 2\pi m/N$ (rad), for positive integers m and N which are not divisible by each other, and θ is the phase angle.

$$x[n+kN] = A\cos\left(\frac{2\pi m}{N}(n+kN) + \theta\right) = A\cos\left(\frac{2\pi m}{N}n + 2\pi mk + \theta\right) = x[n]$$

When sampling

$$x(t) = A\cos(\Omega_0 t + \theta) \qquad -\infty < t < \infty \quad \text{of period } T_0 = 2\pi/\Omega_0,\ \Omega_0 > 0$$

we obtain a periodic discrete sinusoid

$$x[n] = A\cos(\Omega_0 T_s n + \theta) = A\cos\left(\frac{2\pi T_s}{T_0}n + \theta\right)$$

provided that

$$\frac{T_s}{T_0} = \frac{m}{N}, \quad N, m > 0 \ \text{ and not divisible by each other}$$

To avoid frequency aliasing also

$$T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2}$$

Example Is $x[n] = \cos(n + \pi/4)$ obtained by sampling $x(t) = \cos(t + \pi/4)$ with $T_s = 1$ periodic?
if so, indicate its period. Otherwise, find sampling period satisfying Nyquist and when used in sampling $x(t)$ results in periodic signal

$x[n] = x(t)|_{t=nT_s} = \cos(n + \pi/4)$, is not periodic, $\omega = 1$ (rad) cannot be expressed as $2\pi m/N$ for integers m and N (π is irrational)

$x(t)$ has frequency $\Omega = 1$ then Nyquist requires $T_s \leq \frac{\pi}{\Omega} = \pi$
for $x(t)|_{t=nT_s} = \cos(nT_s + \pi/4)$ periodic of period N then

$\cos((n+N)T_s+\pi/4) = \cos(nT_s+\pi/4)$ is necessary that $NT_s = 2k\pi$, k integer

$T_s = 2k\pi/N \leq \pi$ satisfies Nyquist sampling condition and insures the periodicity of $x[nT_s)$

If period $N = 10$, then $T_s = 0.2k\pi$, for k chosen so Nyquist condition is satisfied, i.e.,

$$0 < T_s = k\pi/5 \leq \pi \quad \text{so that} \quad 0 < k \leq 5$$

Choose $k = 1$ and 3 so that N and k are not divisible by each other
If $k = 2$, and 4 would give 5 as the period, and $k = 5$ would give a period of 2 instead of 10

If $k = 1$ then $T_s = 0.2\pi$ satisfies Nyquist sampling rate condition, and

$$x[n] = \cos(0.2n\pi + \pi/4) = \cos\left(\frac{2\pi}{10}n + \frac{\pi}{4}\right)$$

is periodic of period 10. The same for $k = 3$

For discrete-time signal $x[n]$*:*

$$\textbf{Energy:} \qquad \varepsilon_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\textbf{Power:} \qquad P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- $x[n]$ *is* **finite energy** *or* **square summable** *if* $\varepsilon_x < \infty$.
- $x[n]$ *is* **absolutely summable** *if*

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- $x[n]$ *is* **finite power** *if* $P_x < \infty$.

Example "Causal" sinusoid,

$$x[n] = 2\cos(\Omega_0 t - \pi/4)u(t)|_{t=0.1n} = \begin{cases} 2\cos(0.1\Omega_0 n - \pi/4) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Does $x[n]$ have finite energy, finite-power as compared with $x(t)$ when $\Omega_0 = \pi$ and when $\Omega_0 = 3.2$ rad/sec (an upper approximation of π)?

$x(t)$ has infinite energy, and so does $x[n]$, for all Ω_0

$$\varepsilon_x = \sum_{n=-\infty}^{\infty} x[n]^2 = \sum_{n=0}^{\infty} 4\cos^2(0.1\Omega_0 n - \pi/4) \to \infty$$

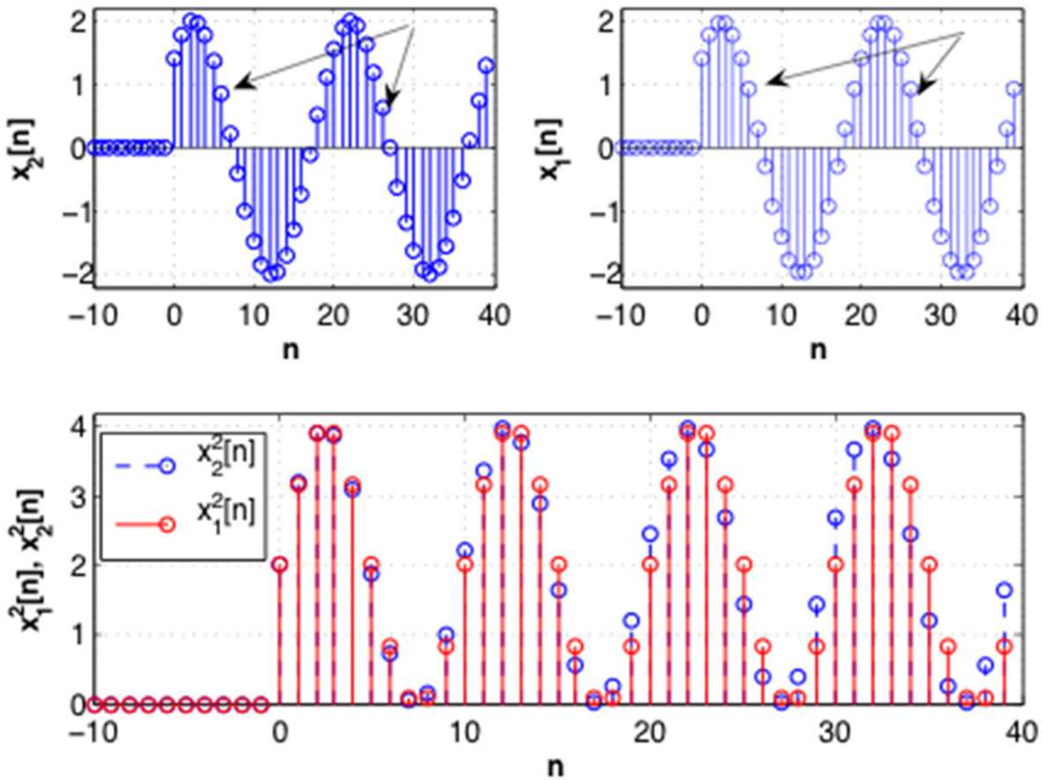
$x(t)$ and $x[n]$ have finite power
(i) $\Omega_0 = \pi$, $x[n] = 2\cos(\pi n/10 - \pi/4) = 2\cos(2\pi n/20 - \pi/4)$ for $n \geq 0$ and zero otherwise
$x[n]$ repeats every $N_0 = 20$ samples for $n \geq 0$,so

$$
\begin{aligned}
P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |x[n]|^2 \\
&= \lim_{N\to\infty} \frac{1}{2N+1}\, N \underbrace{\left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2\right]}_{\text{power of period, } n\geq 0} = \frac{1}{2N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 < \infty \\
&= \frac{4}{40} 0.5 \left[\sum_{n=0}^{10} 1 + \sum_{n=0}^{19} \cos(0.2\pi n - \pi/2)\right] = \frac{2}{40}[20 + 0] = 1
\end{aligned}
$$

(ii) $\Omega_0 = 3.2$, $x[n] = 2\cos(3.2n/10 - \pi/4)$ for $n \geq 0$ and zero otherwise; it does not repeat periodically for $n \geq 0$
$3.2/10$ cannot be expressed as $2\pi m/N$, no close form for the power

$$
P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2
$$

and conjecture that because $x(t)$ has finite power, so would $x[n]$



Even and Odd Discrete-time Signals

A discrete-time signal $x[n]$ is said to be

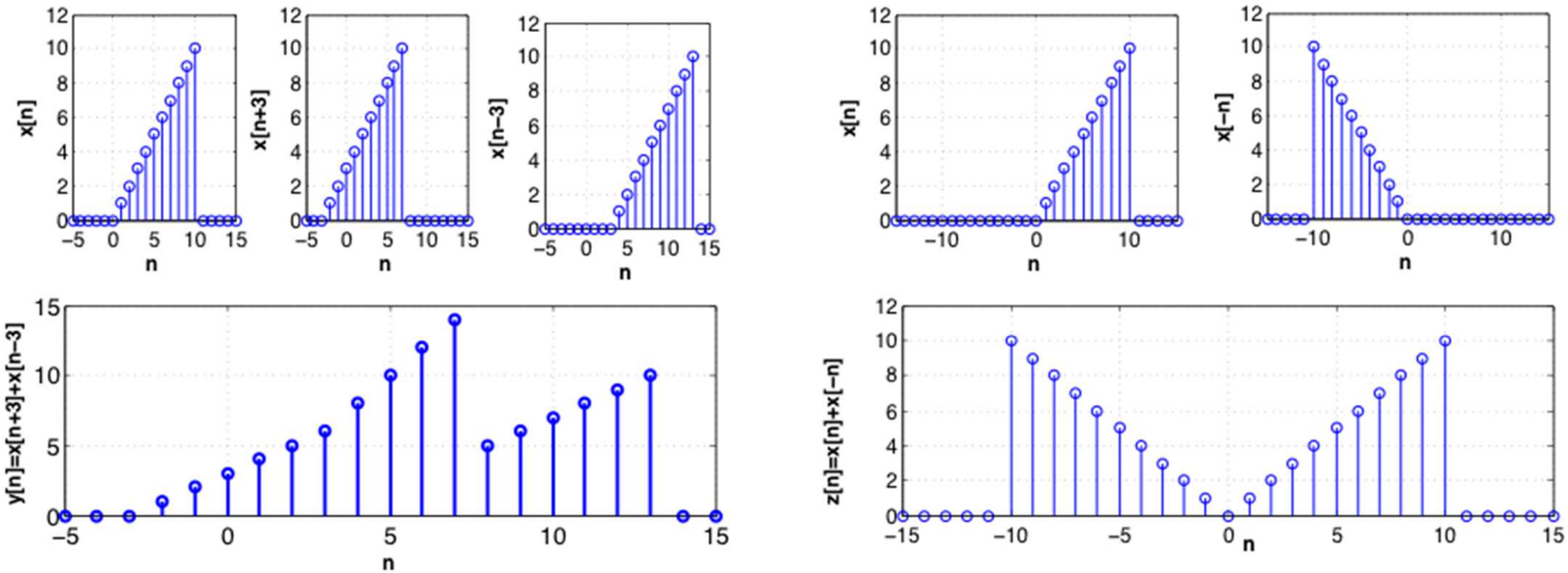
- **delayed** *by N (an integer) samples if $x[n-N]$ is $x[n]$ shifted to the right N samples,*
- **advanced** *by M (an integer) samples if $x[n+M]$ is $x[n]$ shifted to the left M samples,*
- **reflected** *if the variable n in $x[n]$ is negated, i.e., $x[-n]$.*

Even and odd discrete-time signals are defined as

$$x[n] \text{ is } \textbf{even:} \quad \Leftrightarrow \quad x[n] = x[-n]$$
$$x[n] \text{ is } \textbf{odd:} \quad \Leftrightarrow \quad x[n] = -x[-n]$$

Any discrete-time signal $x[n]$ *can be represented as the sum of an even and an odd components*

$$\begin{aligned} x[n] &= \underbrace{\frac{1}{2}\left(x[n] + x[-n]\right)}_{x_e[n]} + \underbrace{\frac{1}{2}\left(x[n] - x[-n]\right)}_{x_o[n]} \\ &= x_e[n] + x_o[n] \end{aligned}$$



Example Find even and odd components of

$$x[n] = \begin{cases} 4-n & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} x_e[n] &= 0.5(x[n] + x[-n]) \\ &= \begin{cases} 2+0.5n & -4 \le n \le -1 \\ 4 & n = 0 \\ 2-0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} x_o[n] &= 0.5(x[n] - x[-n]) \\ &= \begin{cases} -2-0.5n & -4 \le n \le -1 \\ 0 & n = 0 \\ 2-0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Discrete-time Complex Exponential

Given complex numbers $A = |A|e^{j\theta}$ *and* $\alpha = |\alpha|e^{j\omega_0}$, *a* **discrete-time complex exponential** *is a signal of the form*

$$\begin{aligned} x[n] &= A\alpha^n \\ &= |A||\alpha|^n e^{j(\omega_0 n+\theta)} \\ &= |A||\alpha|^n \left[\cos(\omega_0 n+\theta) + j\sin(\omega_0 n+\theta)\right] \end{aligned}$$

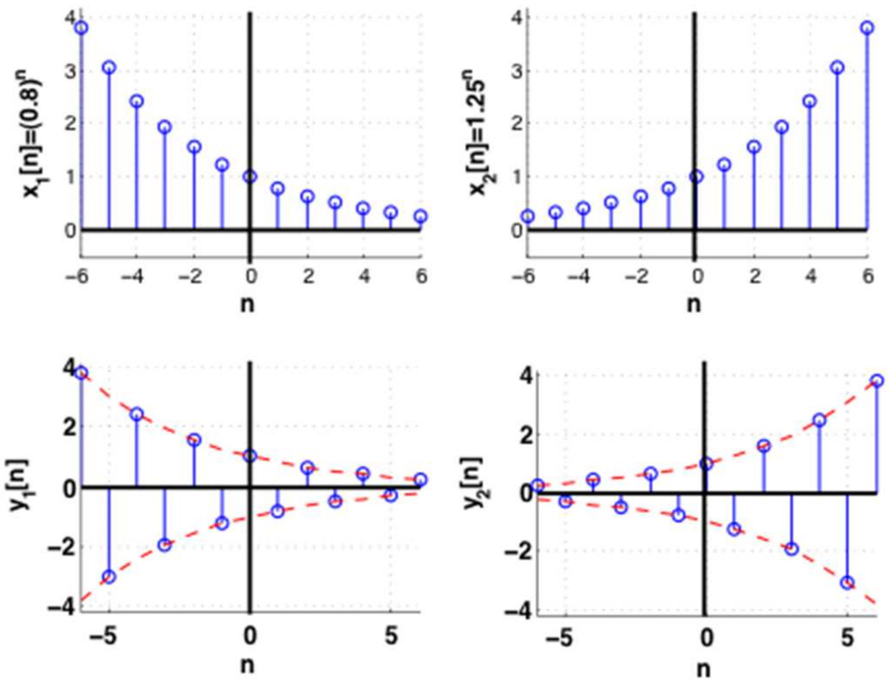
where ω_0 *is a discrete frequency in radians.*

1. Discrete-time vs continuous-time complex exponentials (for simplicity we let A be real) using as sampling period T_s

   Sampling $x(t) = Ae^{(-a+j\Omega_0)t}$, A real
   $x[n] = x(nT_s) = Ae^{(-anT_s+j\Omega_0 nT_s)} = A(e^{-aT_s})^n e^{j(\Omega_0 T_s)n} = A\alpha^n e^{j\omega_0 n}$
   $\alpha = e^{-aT_s}, \quad \omega_0 = \Omega_0 T_s$

2. For $\alpha > 0$ the real exponential

$$x[n] = (-\alpha)^n = (-1)^n\alpha^n = \alpha^n\cos(\pi n)$$



Real exponential $x_1[n] = 0.8^n$, $x_2[n] = 1.25^n$ *(top) and modulated* $y_1[n] = x_1[n]\cos(\pi n)$ *and* $y_2[n] = x_2[n]\cos(\pi n)$

Example Determine $a > 0$, Ω_0 and T_s for

$$x(t) = e^{-at}\cos(\Omega_0 t)u(t)$$

that permit us to obtain a discrete-time signal

$$y[n] = \alpha^n \cos(\omega_0 n) \qquad n \geq 0$$

and zero otherwise by sampling it. If $\alpha = 0.9$ and $\omega_0 = \pi/2$, find a, Ω_0 and T_s that will permit us to obtain $y[n]$ from $x(t)$ by sampling

$$\text{Comparing } x(nT_s) \text{ with } y[n]$$
$$\alpha = e^{-aT_s}$$
$$\omega_0 = \Omega_0 T_s$$

No unique solution. According to Nyquist condition

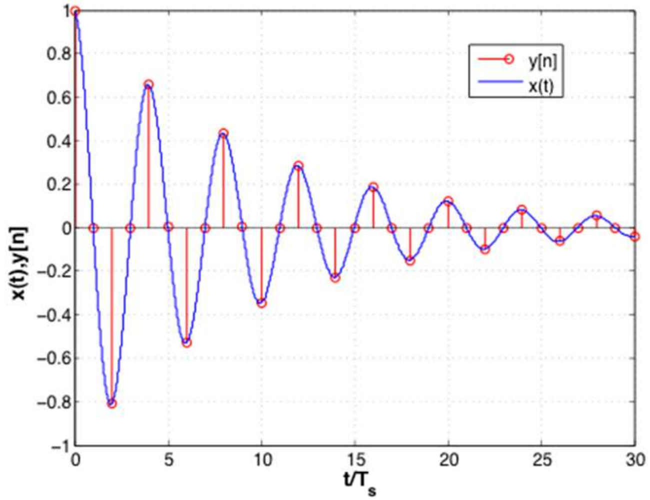
$$T_s \leq \frac{\pi}{\Omega_{max}}$$

If $\Omega_{max} = N\Omega_0$ for $N \geq 2$ (signal is not band-limited so maximum frequency is not known, we are assuming it is a multiple of Ω_0)

$$\text{If } T_s = \pi/N\Omega_0$$
$$\alpha = e^{-a\pi/N\Omega_0}$$
$$\omega_0 = \Omega_0\pi/N\Omega_0 = \pi/N$$

For $\alpha = 0.9$, $\omega_0 = \pi/2$, we have

$$N = 2$$
$$a = -\frac{2\Omega_0}{\pi}\log 0.9$$
$$\text{If } \Omega_0 = 2\pi \quad \Rightarrow \quad a = -4\log 0.9, \quad T_s = 0.25$$



Discrete-time Sinuoids

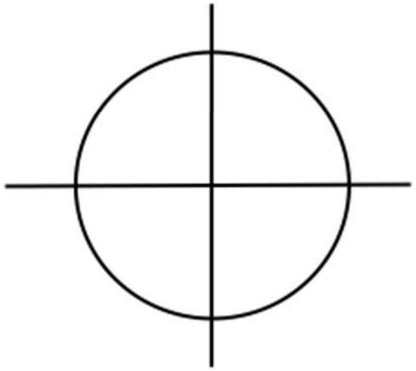
$$x[n] = A\cos(\omega_0 n + \theta) = A\sin(\omega_0 n + \theta + \pi/2) \qquad -\infty < n < \infty$$

If discrete frequency $w_0 = 2\pi m/N$ (rad), for integers m and $N > 0$ which are not divisible $x[n]$ is periodic, oherwise not

Discrete frequency

$\omega = \pi/2, 5\pi/2, 9\pi/2 \cdots = \pi/2$

$\omega = \pi, 3\pi, 5\pi \cdots = \pi = -\pi$



$\omega = 0, 2\pi, 4\pi, \cdots = 0$

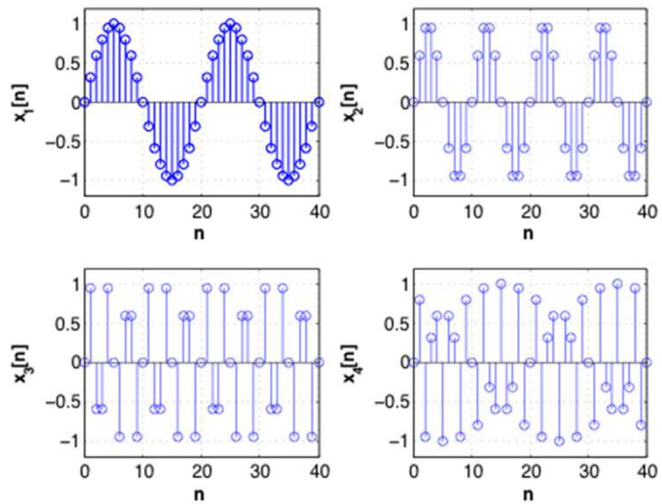
$\omega = 3\pi/2, 7\pi/2, 11\pi/2, \cdots = -\pi/2$

$$x_1[n] = \sin(0.1\pi n) = \sin\left(\frac{2\pi}{20}n\right)$$

$$x_2[n] = \sin(0.2\pi n) = \sin\left(\frac{2\pi}{10}n\right)$$

$$x_3[n] = \sin(0.6\pi n) = \sin\left(\frac{2\pi}{10}3n\right)$$

$$x_4[n] = \sin(0.7\pi n) = \sin\left(\frac{2\pi}{20}7n\right)$$



Discrete-time Unit-step and Unit-sample Signals

The unit-step $u[n]$ and the unit-sample $\delta[n]$ discrete-time signals are defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

These two signals are related as follows

$$\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m] \end{aligned}$$

Generic representation of discrete-time signals

Any discrete-time signal $x[n]$ is represented using unit-sample signals as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Example A train of triangular, discrete-time pulses $t[n]$ of period $N = 11$ has a period

$$\tau[n] = \begin{cases} n & 0 \le n \le 5 \\ -n + 10 & 6 \le n \le 10 \\ 0 & \text{otherwise} \end{cases} .$$

Find then an expression for its finite difference $d[n] = t[n] - t[n-1]$

$$t[n] = \cdots + \tau[n+11] + \tau[n] + \tau[n-11] + \cdots = \sum_{k=-\infty}^{\infty} \tau[n - 11k]$$

The finite difference $d[n]$ is then

$$\begin{aligned} d[n] &= t[n] - t[n-1] \\ &= \sum_{k=-\infty}^{\infty} (\tau[n - 11k] - \tau[n - 1 - 11k]) \end{aligned}$$

The signal $d[n]$ is also periodic of the same period $N = 11$ as $t[n]$. If we let

$$s[n] = \tau[n] - \tau[n-1] = \begin{cases} 1 & 0 \le n \le 5 \\ -1 & 6 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

then

$$d[n] = \sum_{k=-\infty}^{\infty} s[n - 11k]$$

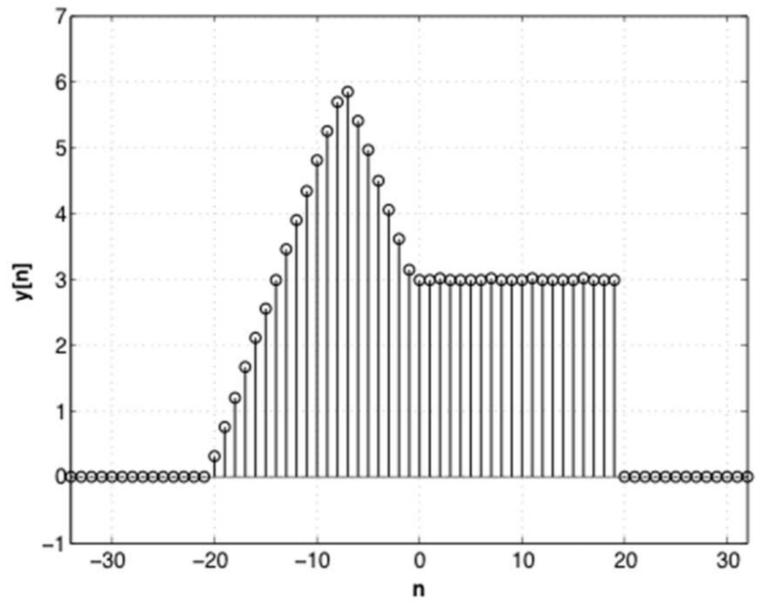
Example Express as function of n the sampled signal

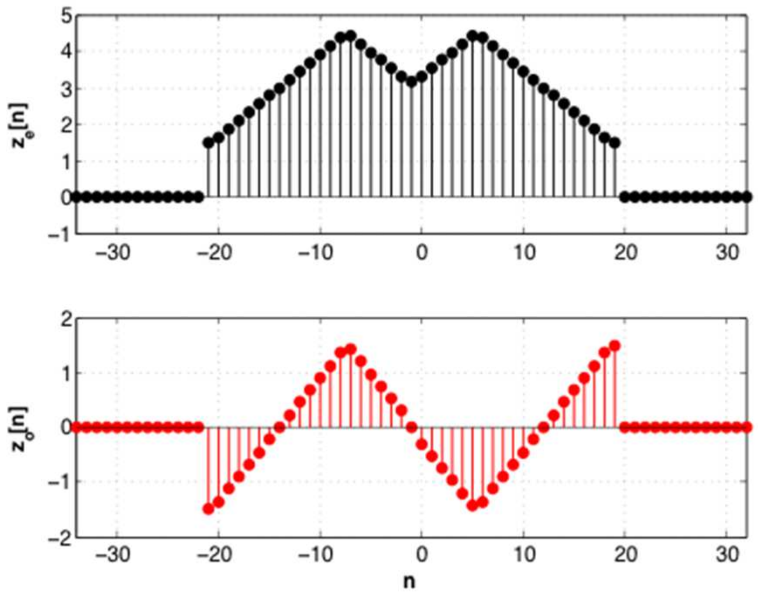
$$y[n] = 3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3)|_{t=0.15n}$$

Use MATLAB to find and plot its even and odd components

$$y(t) = \begin{cases} 0 & t < -3 \\ 3r(t+3) = 3t+9 & -3 \leq t < -1 \\ 3t+9-6r(t+1) = -3t+3 & -1 \leq t < 0 \\ -3t+3+3r(t) = 3 & 0 \leq t < 3 \\ 3-3 = 0 & t \geq 3 \end{cases}$$

$$y[n] = \begin{cases} 0 & n \leq -21 \\ 0.45n+9 & -20 \leq n \leq -6 \\ -0.45n+3 & -7 \leq n \leq 0 \\ 3 & 1 \leq n \leq 19 \\ 0 & n \geq 20 \end{cases}$$





Dynamic system

$$y[n] = \mathcal{S}\{x[n]\} \quad x[n] \text{ input, } y[n] \text{ output}$$

that is

- *Linearity*
- *Time-invariance*
- *Stability*
- *Causality*

A discrete-time system $\mathcal{S}$ *is said to be*

- **Linear:** *if for inputs* $x[n]$ *and* $v[n]$, *and constants* a *and* b, *it satisfies the following*
  - **Scaling** : $\mathcal{S}\{ax[n]\} = a\mathcal{S}\{x[n]\}$
  - **Additivity**: $\mathcal{S}\{x[n] + v[n]\} = \mathcal{S}\{x[n]\} + \mathcal{S}\{v[n]\}$

  or equivalently if **superposition** *applies, i.e.,*

  $$\mathcal{S}\{ax[n] + bv[n]\} = a\mathcal{S}\{x[n]\} + b\mathcal{S}\{v[n]\}$$

- **Time-invariant:** *if for an input* $x[n]$ *with a corresponding output* $y[n] = \mathcal{S}\{x[n]\}$, *the output corresponding to a delayed or advanced version of* $x[n]$, $x[n \pm M]$, *is* $y[n \pm M] = \mathcal{S}\{x[n \pm M]\}$, *for an integer* M.

Example *A Square-root Computation System*
Difference equation, with some initial condition $y[0]$, can be used to find the square root of α:

$$y[n] = 0.5\left[y[n-1] + \frac{\alpha}{y[n-1]}\right] \qquad n > 0$$

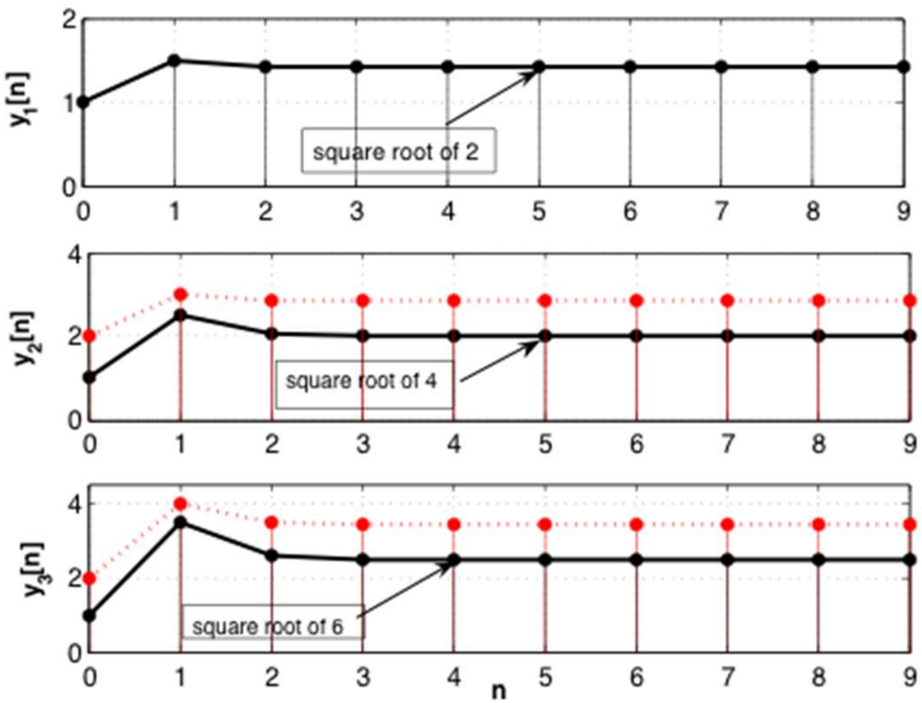
Find recursively solution of this difference equation for $\alpha = 4$ and 2. Is system linear?
Difference equation is first-order, non-linear

$$\begin{aligned}
&\text{Recursive solution}\\
&y[1] = 0.5\left[y[0] + \frac{\alpha}{y[0]}\right]\\
&y[2] = 0.5\left[y[1] + \frac{\alpha}{y[1]}\right]\\
&y[3] = 0.5\left[y[2] + \frac{\alpha}{y[2]}\right]\\
&\quad\vdots
\end{aligned}$$

If $y[0] = 1$, and $\alpha = 4$ (i.e., we wish to find the square root of 4),

$$\begin{aligned}
&y[0] = 1\\
&y[1] = 0.5\left[1 + \frac{4}{1}\right] = 2.5\\
&y[2] = 0.5\left[2.5 + \frac{4}{2.5}\right] = 2.05\\
&\quad\vdots
\end{aligned}$$

converging to 2. When $n \to \infty$ then $y[n] = y[n-1] = Y$, so $Y = 0.5Y + 0.5(4/Y)$ or $Y = \sqrt{4} = 2$



Non-linear system: square root of 2 (top), square root of 4 compared with twice the square root of 2, (bottom) sum of previous responses with response when computing square root of $2 + 4$. Middle figure shows scaling does not hold and the bottom figure that additivity does not hold, either. System is non-linear.

Depending on the relation between the input $x[n]$ and the output $y[n]$ two types of discrete-time systems of interest are:

- **Recursive system**

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m] \qquad n \geq 0$$

$$\textit{initial conditions} \;\; y[-k], \;\; k = 1, \cdots, N-1$$

This system is also called **infinite impulse response (IIR)**.

- **Non-recursive system**

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m]$$

This system is also called **finite impulse response (FIR).**

Example *Moving-average Discrete Filter*
Show that the third-order moving-average FIR filter (also called a **smoother**)

$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$ with input $x[n]$ and output $y[n]$ is LTI

Linearity — Let input be $ax_1[n] + bx_2[n]$, and $\{y_i[n], \; i = 1, 2\}$ are the corresponding outputs to $\{x_i[n], \; i = 1, 2\}$, then

$$\frac{1}{3}[(ax_1[n] + bx_2[n]) + (ax_1[n-1] + bx_2[n-1]) + (ax_1[n-2] + bx_2[n-2])] = ay_1[n] + by_2[n]$$

thus linear.

Time invariance — If input is $x_1[n] = x[n-N]$ the corresponding output to it is

$$\begin{aligned} \frac{1}{3}(x_1[n] + x_1[n-1] + x_1[n-2]) &= \frac{1}{3}(x[n-N] + x[n-N-1] + x[n-N-2]) \\ &= y[n-N] \end{aligned}$$

i.e., the system is time-invariant.

- Recursive systems are represented by **difference equation**

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=1}^{M-1} b_m x[n-m], \qquad n \geq 0, \text{ ICs } y[-k], \; k = 1, \cdots, N-1$$

- Difference equation characterizes dynamics of discrete systems
- Difference equation could be approximation of differential equation representing continuous-time system
- Complete response

$$\begin{aligned} y[n] &= y_{zi}[n] + y_{zs}[n], \; y_{zi}[n] \text{ zero-input response}, \; y_{zs}[n] \text{ zero-state response} \\ &= y_t[n] + y_{ss}[n], \; y_t[n] \text{ transient response}, \; y_{ss}[n] \text{ steady-state response} \end{aligned}$$

Convolution Sum

Let $h[n]$ be the impulse response of an LTI discrete-time system, or the output of the system corresponding to an impulse $\delta[n]$ as input, and initial conditions (if needed) equal to zero.
Using the generic representation of the input $x[n]$ of the LTI system

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

the output of the system is given by either of the following two forms of the **convolution sum**:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \end{aligned}$$

Remarks

- The output of an FIR systems is the convolution sum of the input and the impulse response of the system

$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k]$$

$$\text{impulse response} \quad h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_{N-1}\delta[n-(N-1)]$$

$$h[n] = b_n, \quad n = 0, \cdots, N-1 \qquad \Rightarrow \qquad y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

- Convolution sum as an operator is linear and commutative

$$\begin{aligned} [h * x][n] &= \sum_k x[k]h[n-k] = \sum_k x[n-k]h[k] \\ &= [x * h][n] \end{aligned}$$

- Just as with analog systems, when conecting two LTI discrete-time systems (with impulse responses $h_1[n]$ and $h_2[n]$) in series or in parallel, their respective impulse responses are given by $[h_1 * h_2][n]$ and $h_1[n] + h_2[n]$

Example Moving-averaging filter

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$
$$\text{input } \; x[n], \quad \text{output } y[n]$$

Find the impulse response $h[n]$ of this filter. Then for

1. $x[n] = u[n]$, find $y[n]$ using the input-output relation and the convolution sum.

2. $x[n] = A\cos(2\pi n/N)u[n]$, determine the values of A, and N, so that the steady state response of the filter is zero

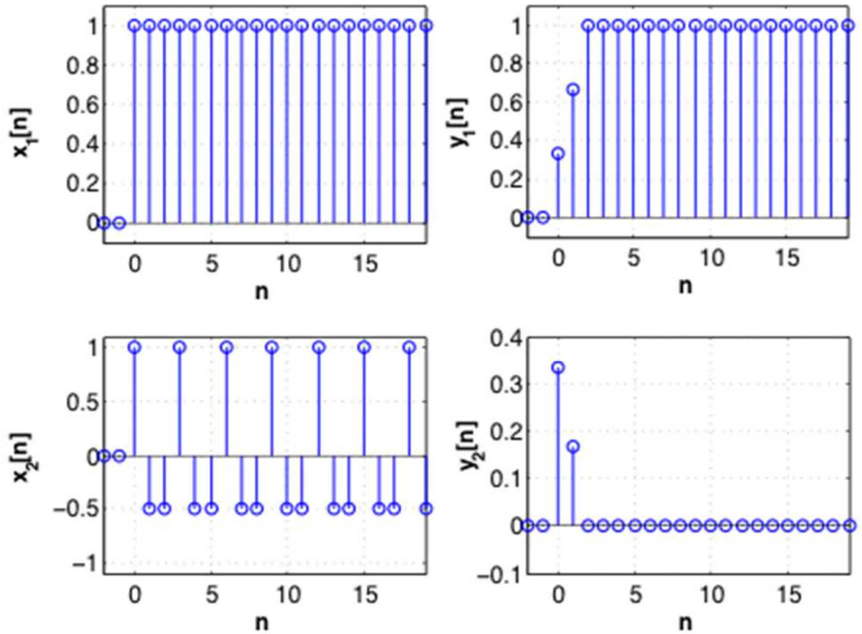
(1) $x[n] = \delta[n]$, $y[n] = h[n]$, no initial conditions are needed

$$h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$

The convolution sum coincides with the input/output equation. This holds for any FIR filter.

$$\begin{aligned}
y[n] &= 0, \; n < 0 \\
y[0] &= \frac{1}{3}(x[0] + x[-1] + x[-2]) = \frac{1}{3}x[0] \\
y[1] &= \frac{1}{3}(x[1] + x[0] + x[-1]) = \frac{1}{3}(x[0] + x[1]) \\
y[2] &= \frac{1}{3}(x[2] + x[1] + x[0]) = \frac{1}{3}(x[0] + x[1] + x[2]) \\
&\dots
\end{aligned}$$

If $x[n] = u[n]$ then we have that $y[0] = 1/3$, $y[1] = 2/3$ and $y[n] = 1$ for $n \geq 2$.
(2) For $n \geq 2$, $y[n]$ is the average of the present and past two values of the input. If $x[n] = A\cos(2\pi n/N)$, if we let $N = 3$, and A be any real value the input repeats every 3 samples and the local average of 3 of its values is zero, giving $y[n] = 0$ for $n \geq 2$

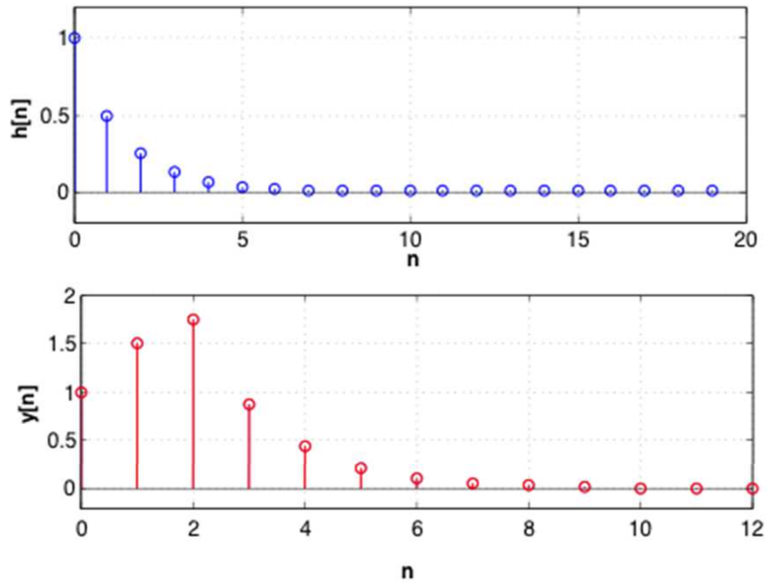


Example. Autoregressive system

$$y[n] = 0.5y[n-1] + x[n] \qquad n \geq 0$$

Find the impulse response $h[n]$ of the system and then compute the response of the system to $x[n] = u[n] - u[n-3]$ using the convolution sum. Verify results with MATLAB.

```
a=[1 -0.5];b=1;  % coefficients of the difference equation
d=[1 zeros(1,99)];  % approximate delta function
h=filter(b,a,d);        % impulse response
x=[ones(1,3) zeros(1,10)];  % input
y=filter(b,a,x);  % output from filter function
y1=conv(h,x); y1=y1(1:length(y))  % output from conv
```

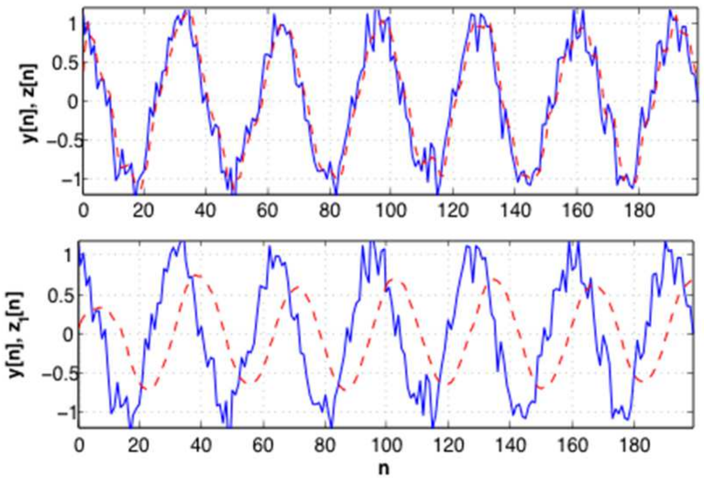


Linear and Non-linear Filtering

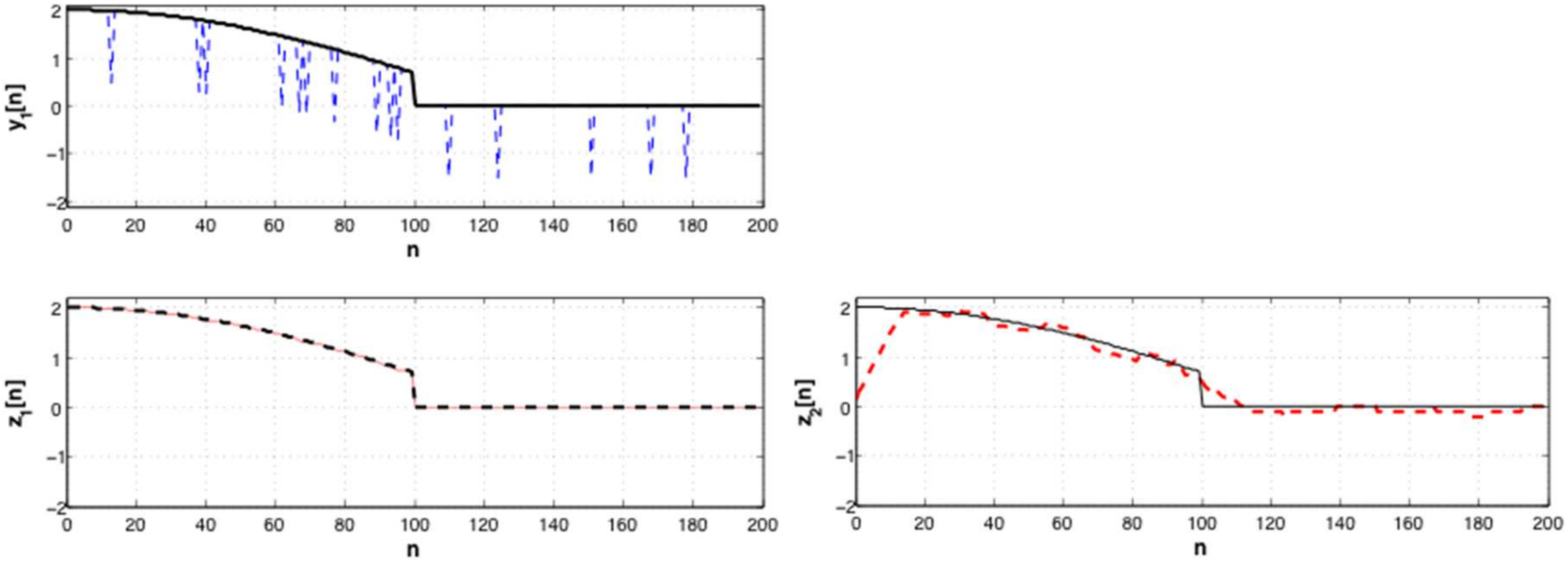
Linear filtering

$$y[n] = x[n] + \eta[n], \text{ where } x[n] \text{ desired signal and } \eta[n] \text{ noise}$$

$$z[n] = \frac{1}{M} \sum_{k=0}^{M-1} y[n-k] \text{ averaging filter}$$



Averaging filtering with filters of order $M = 3$ (top figure), and of order $M = 15$ result (bottom figure) used to get rid of Gaussian noise added to a sinusoid $x[n] = \cos(\pi n/16)$. Solid line corresponds to the noisy signal, while the dashed line is for the filtered signal. The filtered signal is very much like the noisy signal (see top figure) when $M = 3$ is the order of the filter, while the filtered signal looks like the sinusoid, but shifted, when $M = 15$.



Non-linear 5^{th}-order median filtering (bottom left) versus linear 15^{th}-order averager (bottom right) corresponding to the noisy signal (dashed line) and clean signal (solid line) on top. Clean signal (solid line) is superposed on de-noised signal (dashed line) in the bottom figures.

Causality of Discrete-time Systems

- Real-time processing requires causality
- If data is stored no causality is needed

A discrete-time system $\mathcal{S}$ is **causal** *if:*

- *whenever the input $x[n] = 0$, and there are no initial conditions, the output is $y[n] = 0$,*
- *the output $y[n]$ does not depend on future inputs.*

- *An LTI discrete-time system is* **causal** *if the impulse response of the system is such that*

$$h[n] = 0 \qquad n < 0$$

- *A signal $x[n]$ is said to be* **causal** *if*

$$x[n] = 0 \qquad n < 0$$

- *For a causal LTI discrete-time system with a causal input $x[n]$ its output $y[n]$ is given by*

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] \qquad n \geq 0$$

where the lower limit of the sum depends on the input causality, $x[k] = 0$ for $k < 0$, and the upper limit on the causality of the system, $h[n-k] = 0$ for $n-k < 0$ or $k > n$.

BIBO Stability of LTI Discrete-time Systems

LTI system represented by convolution sum

$$|y[n]| \leq \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq M \sum_{k=-\infty}^{\infty} |h[k]| \leq ML < \infty$$

An LTI discrete-time system is said to be BIBO stable if its impulse response $h[n]$ is absolutely summable

$$\sum_{k} |h[k]| < \infty$$

Example Deconvolution: assume the input $x[n]$ and the output $y[n]$ of a causal LTI system are given, find equations to compute recursively the impulse response $h[n]$ of the system. Consider finding the impulse response $h[n]$ of a causal LTI system with input $x[n] = u[n]$ and output $y[n] = \delta[n]$. Use the MATLAB *deconv* to find $h[n]$.

System is causal and LTI

$$y[n] = \sum_{m=0}^{n} h[n-m]x[m] = h[n]x[0] + \sum_{m=1}^{n} h[n-m]x[m]$$

$$h[n] = \frac{1}{x[0]}\left[y[n] - \sum_{m=1}^{n} h[n-m]x[m]\right]$$

$$h[0] = \frac{1}{x[0]}y[0]$$

$$h[1] = \frac{1}{x[0]}\left(y[1] - h[0]x[1]\right)$$

$$h[2] = \frac{1}{x[0]}\left(y[2] - h[0]x[2] - h[1]x[1]\right)$$

$$\vdots$$

When $y[n] = \delta[n]$ and $x[n] = u[n]$

$$h[0] = \frac{1}{x[0]}y[0] = 1$$

$$h[1] = \frac{1}{x[0]}\left(y[1] - h[0]x[1]\right) = 0 - 1 = -1$$

$$h[2] = \frac{1}{x[0]}\left(y[2] - h[0]x[2] - h[1]x[1]\right) = 0 - 1 + 1 = 0$$

$$h[3] = \frac{1}{x[0]}\left(y[3] - h[0]x[3] - h[1]x[2] - h[2]x[3]\right) = 0 - 1 + 1 - 0 = 0$$

$$\vdots$$

and in general $h[n] = \delta[n] - \delta[n-1]$.
When using *deconv* make sure that the length of $y[n]$ is always larger than that of $x[n]$

```
% Deconvolution
clear all
x=ones(1,100);
y=[1 zeros(1,100)]; % (a) correct h
% y=[1 zeros(1,99)]; % (b) incorrect h
[h,r]=deconv(y,x)
```

Remarks

1. Non-recursive or FIR systems are BIBO stable, as their impulse responses are of finite length and absolutely summable.
2. For a recursive or IIR system represented by a difference equation, to established stability we need to find the system impulse response $h[n]$ and determine whether it is absolutely summable or not.
3. A much simpler way to test the stability of an IIR system will be based on the location of the poles of the Z-transform of $h[n]$, as we will see in the next chapter.

Example Consider an autoregressive system

$$y[n] = 0.5y[n-1] + x[n]$$

Determine if the system is BIBO stable.
The impulse response of the system is $h[n] = 0.5^n u[n]$, checking the BIBO stability condition

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$$

thus the system is BIBO stable.

What have we accomplished?

- Obtained similar theory for discrete- and continuous-time signals and systems with distinct differences
  - Sampling period determines radian frequency of discrete signals
- Discrete frequency is finite but circular
- Discrete sinusoids not necessarily periodic

Where do we go from here?

- Z-transform analysis
  - Z-transform relation to Laplace transform
  - Discrete-time Fourier transform and relation with Z-transform