Signals and Systems Using MATLAB

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Chapter 8 - Sampling Theory

What is in this chapter?

- Uniform sampling
  - Band-limited signals and Nyquist condition
  - Signal reconstruction
- Practical aspects of sampling

Ideal Impulse Sampling

Sampling $x(t)$ at uniform times $\{nT_s\}$ gives a sampled signal

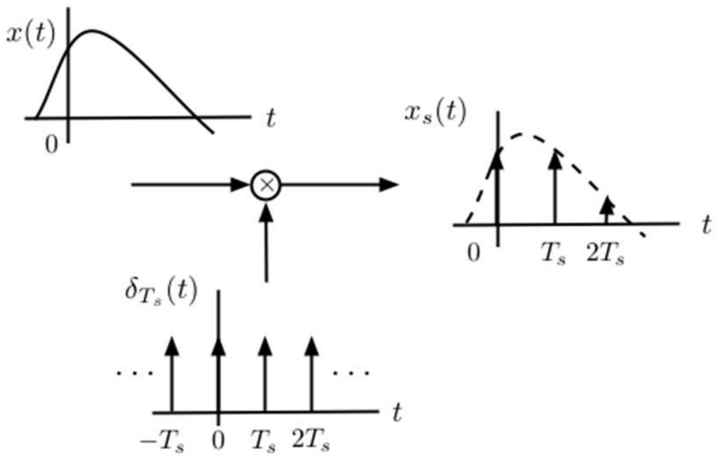
$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

or a sequence of samples $\{x(nT_s)\}$
Sampling is equivalent to modulating the sampling signal

$$\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$$

periodic of period T_s (the sampling period) with $x(t)$.
If $X(\Omega) = \mathcal{F}[x(t)]$ then

$$\begin{aligned} X_s(\Omega) &= \mathcal{F}[x_s(t)] \\ &= \frac{1}{T_s}\sum_k X(\Omega - k\Omega_s) \\ &= \sum_n x(nT_s)e^{-j\Omega T_s n}, \qquad \Omega_s = \frac{2\pi}{T_s}. \end{aligned}$$

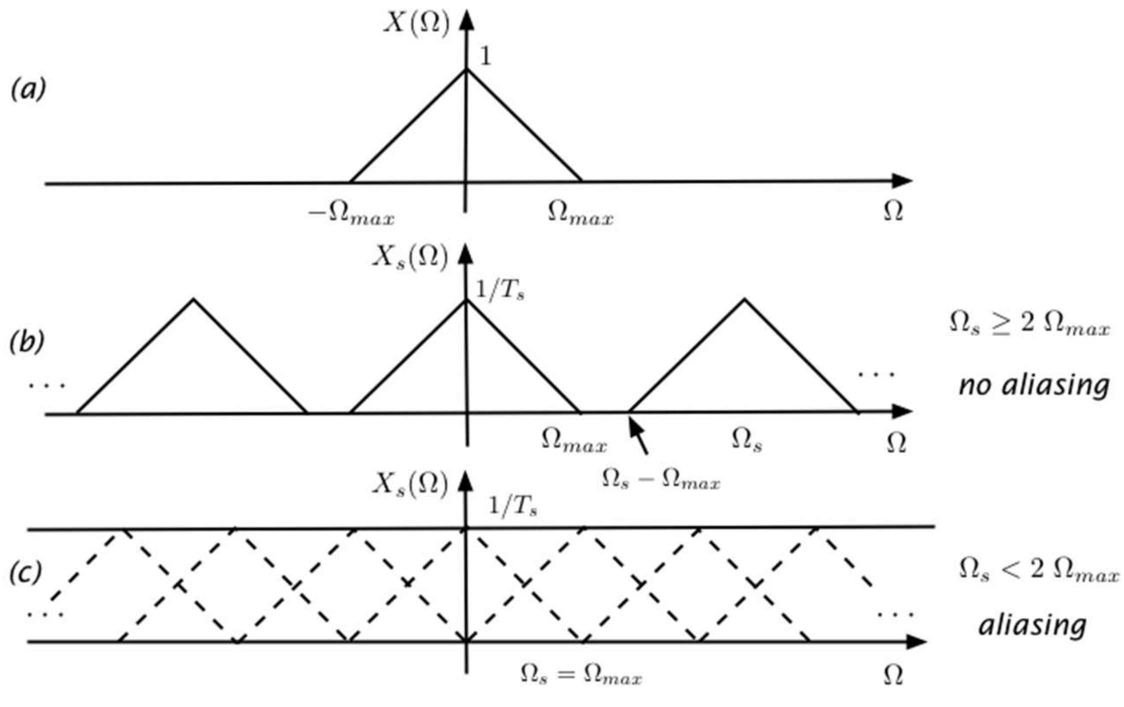


Band-limited signal
$x(t)$ is band-limited if it has low-pass spectrum of finite support, i.e.,

$$X(\Omega) = 0 \quad |\Omega| > \Omega_{max}$$

Ω_{max} maximum frequency in $x(t)$

Nyquist sampling rate
Choose Ω_s so that the spectrum of the sampled signal consists of shifted non-overlapping versions of $(1/Ts)X(\Omega)$ or $\Omega_s \geq 2\Omega_{max}$



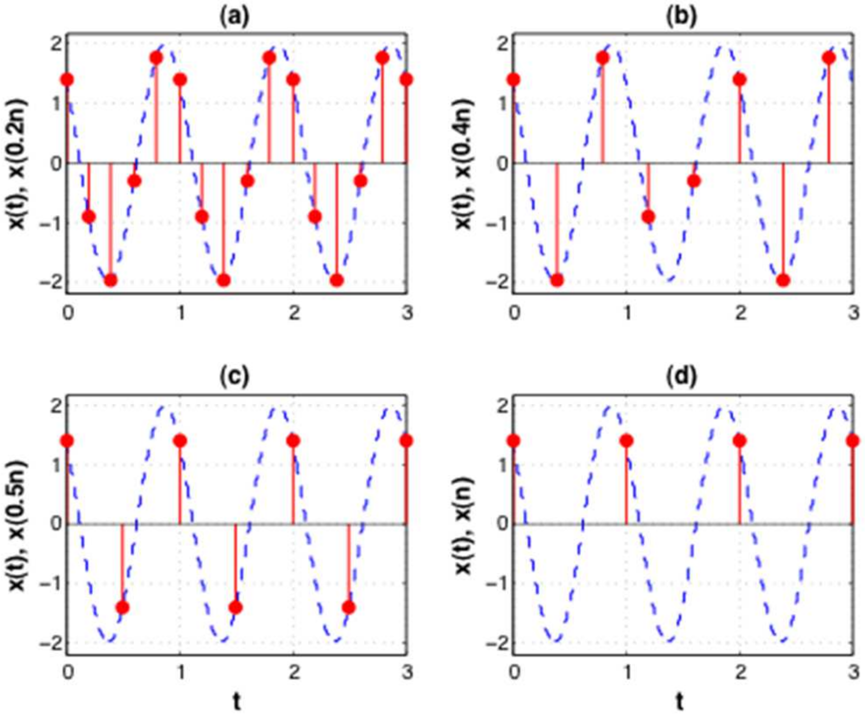
(a) Spectrum of band-limited signal, (b) spectrum of sampled signal when satisfying the Nyquist sampling rate, (c) spectrum of sampled signal with aliasing (superposition of spectra, shown in dashed lines, gives a constant shown by continuous line).

Example Is $x(t) = 2\cos(2\pi t + \pi/4)$, $-\infty < t < \infty$ bandlimited? For $T_s = 0.4$, 0.5 and 1 sec/sample is Nyquist sampling rate satisfied?

$x(t)$ only has frequency 2π, so it is bandlimited with $\Omega_{max} = 2\pi$ (rad/sec)
For any T_s, sampled signal:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \qquad T_s \text{ sec/sample}$$

$$x(nT_s) = x(t)|_{t=nT_s}$$



- $T_s = 0.4$ sec/sample, sampling frequency (rad/sec) $\Omega_s = 2\pi/T_s = 5\pi > 2\Omega_{max} = 4\pi$, Nyquist sampling rate condition satisfied, 3 samples per period (no loss of information – no aliasing)

- $T_s = 0.5$ sec/sample, sampling frequency (rad/sec) $\Omega_s = 2\pi/T_s = 4\pi = 2\Omega_{max}$, barely satisfying the Nyquist sampling rate, 2 samples per period

- $T_s = 1$ sec/sample, sampling frequency (rad/sec) $\Omega_s = 2\pi/T_s = 2\pi < 2\Omega_{max}$, Nyquist sampling rate condition is not satisfied (loss of information – aliasing)

Example Is $x_1(t) = u(t + 0.5) - u(t - 0.5)$ band-limited? If not, determine an approximate maximum frequency

$x_1(t) = u(t + 0.5) - u(t - 0.5)$ can be sampled with $T_s << 1$, e.g., $T_s = 0.01$ sec/sample giving

$$\text{discrete–time signal } \; x_1(nT_s) = 1, \; 0 \leq nT_s = 0.01n \leq 1 \;\text{ or }\; 0 \leq n \leq 100$$

But, $x_1(t)$ is not band-limited

$$X_1(\Omega) = \frac{e^{j0.5\Omega} - e^{-j0.5\Omega}}{j\Omega} = \frac{\sin(0.5\Omega)}{0.5\Omega} \;\text{ has no maximum frequency}$$

Parseval's energy relation

$$\begin{aligned}
&E_{x_1} = 1 \;\text{ the area under }\; x_1^2(t) \\
&\text{find } \;\Omega_M\; \text{ such that } \; .99E_{x_1} \text{ in } [-\Omega_M, \Omega_M] \\
&0.99 = \frac{1}{2\pi}\int_{-\Omega_M}^{\Omega_M} \left[\frac{\sin(0.5\Omega)}{0.5\Omega}\right]^2 d\Omega
\end{aligned}$$

Using MATLAB $\Omega_M = 20\pi$ so $T_s < \pi/\Omega_M = 0.05$ sec/sample

The Nyquist-Shannon Sampling Theorem

If a low-pass continuous-time signal $x(t)$ *is band-limited (i.e., it has a spectrum* $X(\Omega)$ *such that* $X(\Omega) = 0$ *for* $|\Omega| > \Omega_{max}$*, where* Ω_{max} *is the maximum frequency in* $x(t)$*) we then have:*

- $x(t)$ *is uniquely determined by its samples* $x(nT_s) = x(t)|_{t=nT_s}$, $n = 0, \pm 1, \pm 2, \cdots$, *provided that the sampling frequency* Ω_s *(rad/sec) is such that*

  $$\Omega_s \geq 2\Omega_{max} \qquad \text{Nyquist sampling rate condition}$$

  or equivalently if the sampling rate f_s *(samples/sec) or the sampling period* T_s *(sec/sample) are given by*

  $$f_s = \frac{1}{T_s} \geq \frac{\Omega_{max}}{\pi}$$

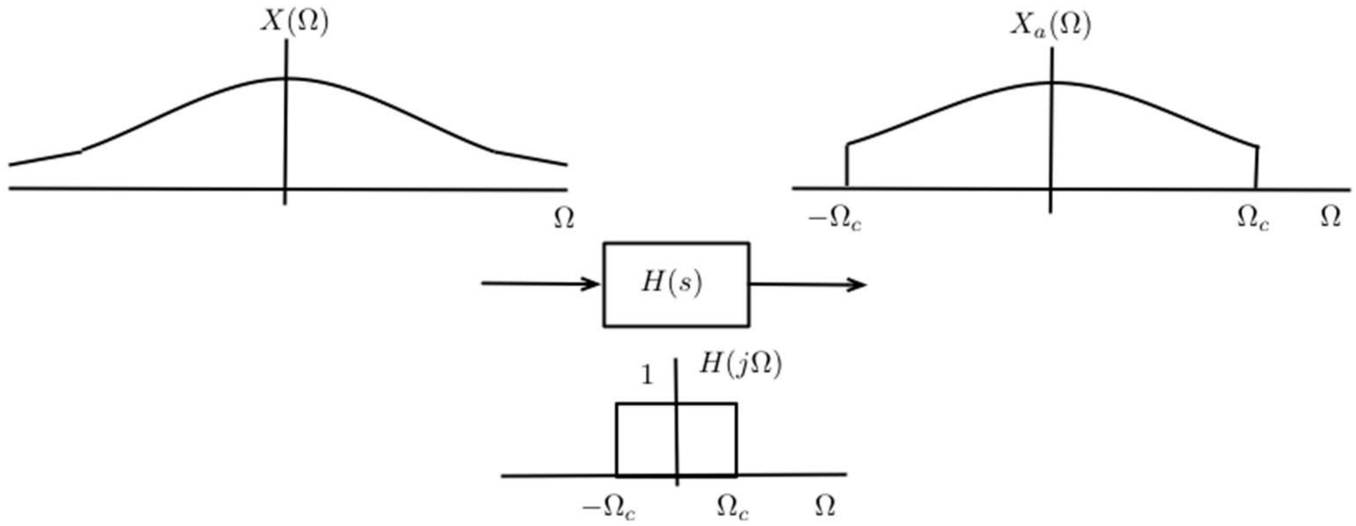
- *When the Nyquist sampling rate condition is satisfied, the original signal* $x(t)$ *can be reconstructed by passing the sampled signal* $x_s(t)$ *through an ideal low-pass filter with the following frequency response:*

  $$H(\Omega) = \begin{cases} T_s & \frac{-\Omega_s}{2} < \Omega < \frac{\Omega_s}{2} \\ 0 & \text{elsewhere} \end{cases}$$

  The reconstructed signal is given by the following sinc interpolation from the samples

  $$x_r(t) = \sum_n x(nT_s) \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}.$$

Antialiasing filtering



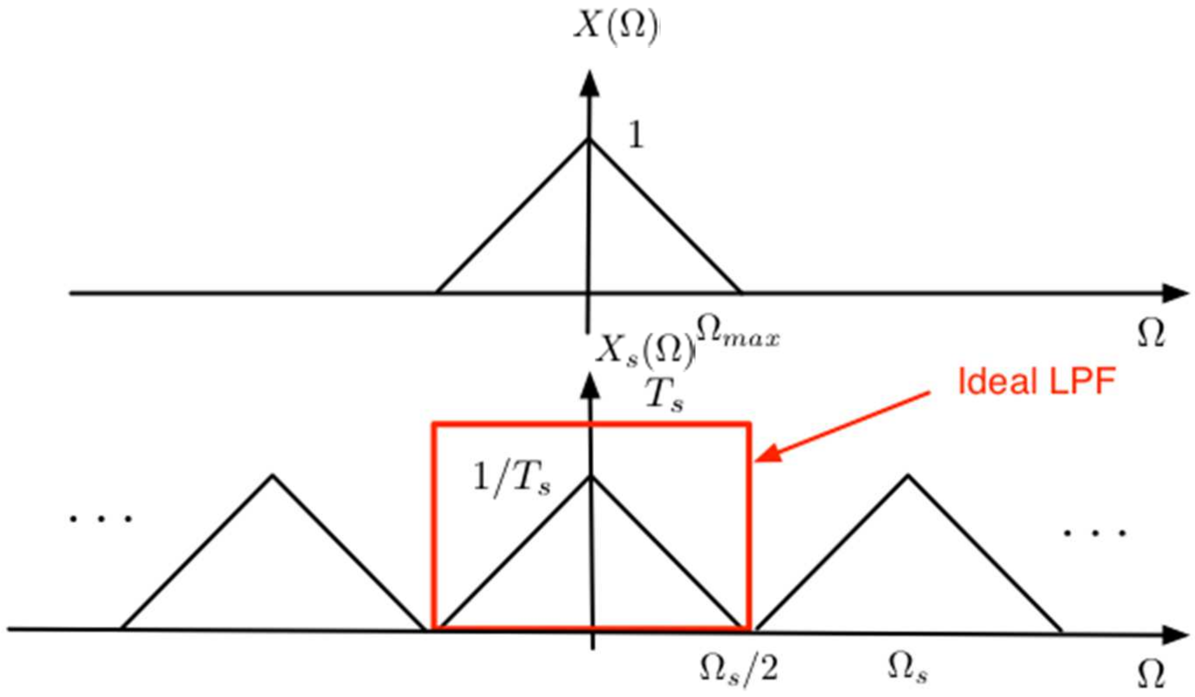
- Anti-aliasing filtering: If signal is not band-limited, pass it through an ideal low-pass filter to get band-limited output (max frequency = cutoff frequency of filter)
- Ouput of anti-alising filter is smoothed version of the original signal — high frequencies of the signal have been removed
- In applications, cut-off frequency of the antialiasing filter set according to prior knowledge, e.g.,
  - sampling speech: frequency band $[100, 5000]$ Hz provides understandable speech in phone conversations $\Rightarrow$ cut-off frequency 5KHz, $f_s = 10,000$ samples/sec
  - sampling music: frequency band $[0, 22,000]$ Hz provides music with good fidelity $\Rightarrow$ cut-off 22KHz, $f_s = 44,000$ samples/sec

If $x(t)$ band-limited, $X(\Omega)$ with maximum frequency Ω_{max}, if $\Omega_s > 2\Omega_{max}$, $X_s(\Omega)$ is superposition of shifted versions of the spectrum $X(\Omega)$, multiplied by $1/T_s$, with no overlaps $\Rightarrow$ $x(t)$ can be recovered from $x_s(t)$ by low-pass filtering

$$\text{Ideal low-pass analog filter} \quad H_{lp}(\Omega) = \begin{cases} T_s & -\Omega_s/2 < \Omega < \Omega_s/2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Filter output} \quad X_r(\Omega) = \begin{cases} X(\Omega) & -\Omega_s/2 < \Omega < \Omega_s/2 \quad \text{where} \quad \Omega_s/2 = \Omega_{max} \\ 0 & \text{elsewhere} \end{cases}$$

coincides with $X(\Omega)$ so $x(t)$ is recovered



$H_{lp}(s)$ ideal LPF

$$h_{lp}(t) = \frac{T_s}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t} d\Omega = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$$

reconstructed signal

$$x_r(t) = [x_s * h_{lp}](t) = \int_{-\infty}^{\infty} x_s(\tau) h_{lp}(t-\tau) d\tau = \sum_n x(nT_s) \frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s}$$

$x_r(t)$ an interpolation in terms of time-shifted sinc signals with amplitudes the samples $\{x(nT_s)\}$

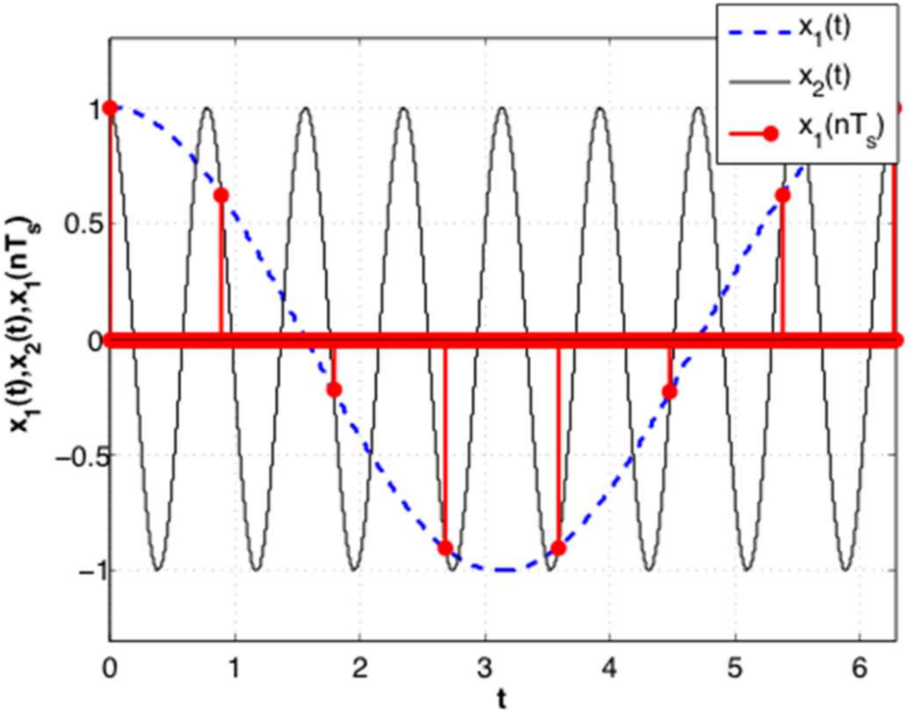
Example Sample following sinusoids $T_s = 2\pi/\Omega_s$

$$\begin{aligned} x_1(t) &= \cos(\Omega_0 t) \qquad -\infty \leq t \leq \infty \\ x_2(t) &= \cos((\Omega_0 + \Omega_s)t) \qquad -\infty \leq t \leq \infty \end{aligned}$$

$$\begin{aligned} x_1(nT_s) &= \cos(\Omega_0 nT_s) \qquad -\infty \leq n \leq \infty \\ x_2(nT_s) &= \cos((\Omega_0 + \Omega_s)nT_s) \qquad -\infty \leq n \leq \infty \end{aligned}$$

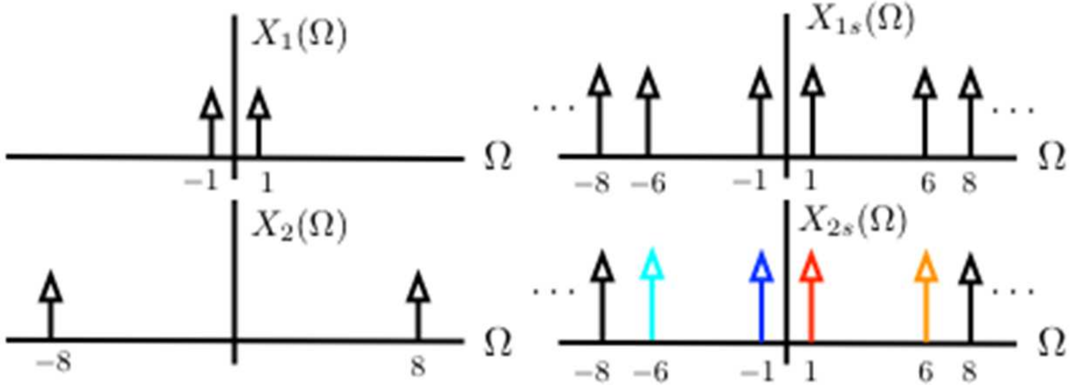
but since $\Omega_s T_s = 2\pi$ the sinusoid $x_2(nT_s)$ can be written

$$x_2(nT_s) = \cos((\Omega_0 T_s + 2\pi)n) = \cos(\Omega_0 T_s n) = x_1(nT_s)$$



$$x_1(t) = \cos(t) \quad x_2(t) = \cos((7+1)t)$$

$$\Omega_s = 7 > 2\Omega_0 = 2$$



$$\Omega_s = 7 > 2(\Omega_0 + \Omega_s) = 2 * 8 = 16$$

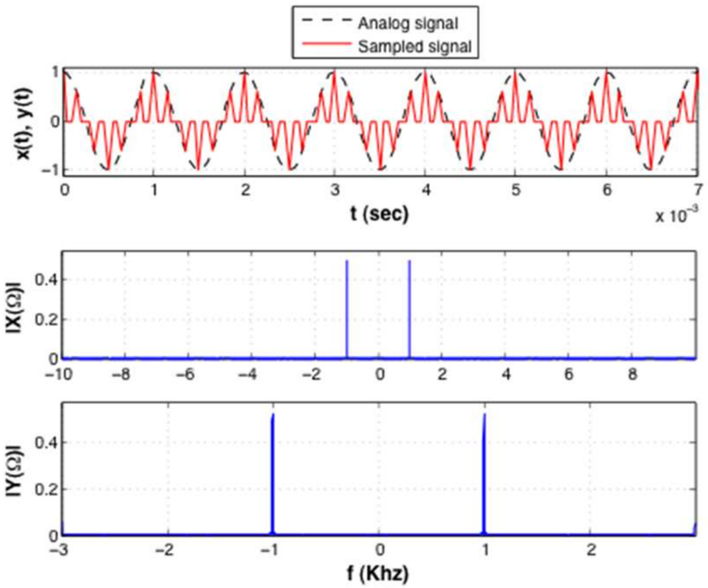
Sampling Simulation with MATLAB

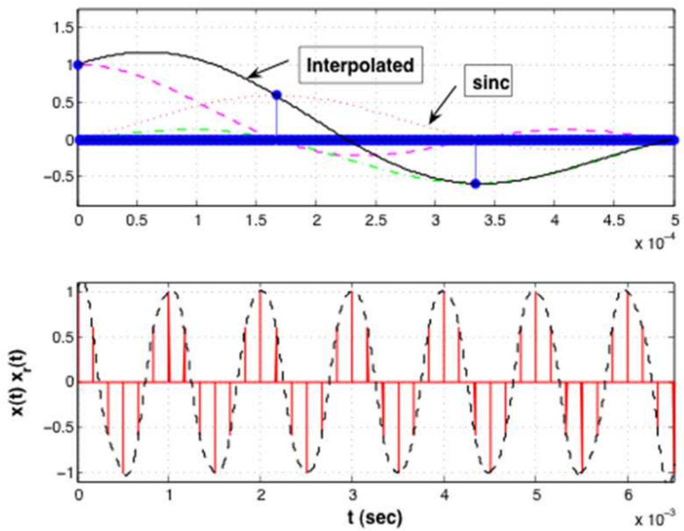
Problems

- Representation of analog signals: use two sampling rates: one under study, f_s, one to simulate the analog signal, $f_{sim} >> f_s$
- Computation of the analog Fourier transform of $x(t)$: approximate it with fast Fourier transform (FFT) multiplied by the sampling period

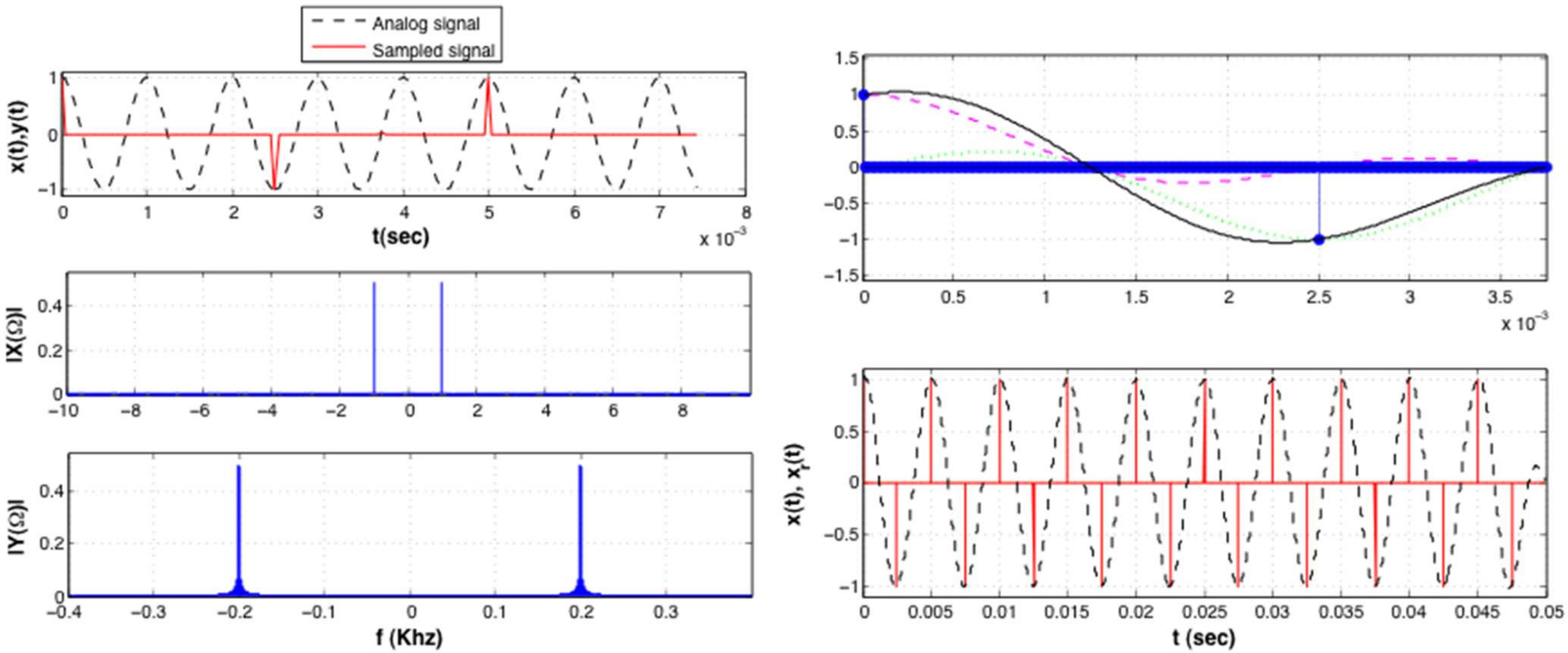
Sampling a sinusoid $x(t) = \cos(2\pi f_0 t)$, $f_0 = 1,000$, using simulation sampling frequency $f_{sim} = 20,000$ samples/sec

No aliasing sampling — sample $x(t)$ with $f_s = 6,000 > 2f_0 = 2,000$, $|X(\Omega)|$ corresponds to $x(t)$, while $|Y(\Omega)|$ is first period of the spectrum of the sampled signal (spectrum of the sampled signal is periodic of period $\Omega_s = 2\pi f_s$)





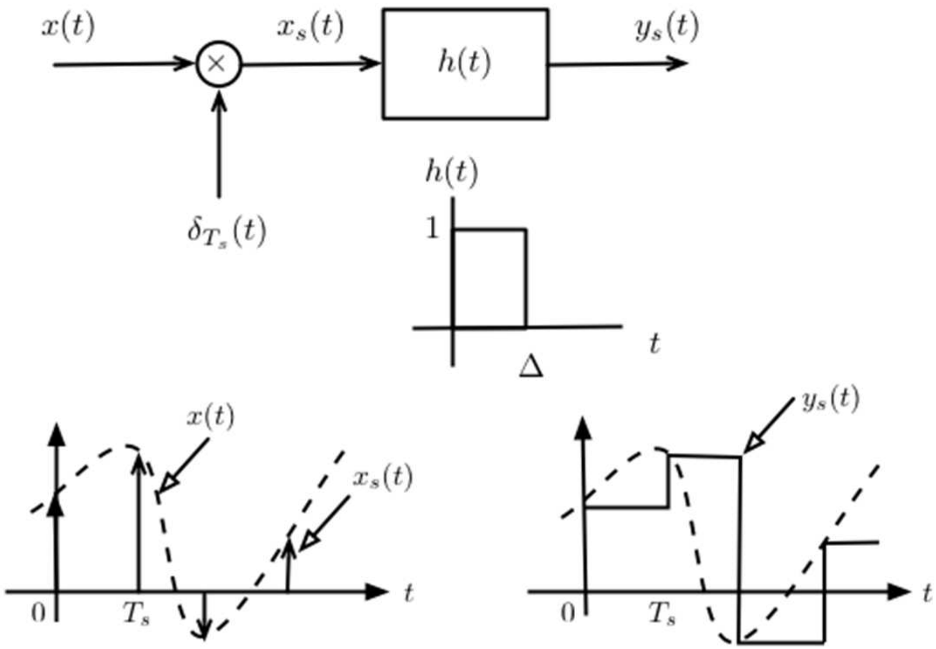
Sampling with aliasing — sample $x(t)$ with $f_s = 800 < 2f_s = 2,000$, $|X(\Omega)|$ same as before, $|Y(\Omega)|$ which is a period of the spectrum of the sampled signal $y(t)$ displays a frequency of 200 Hz



Practical Aspects of Sampling

- Analog to digital and digital to analog conversions are done by A/D and D/A converters
- Difference with ideal versions
  - pulses rather than impulses
  - quantization and coding

Sample and Hold



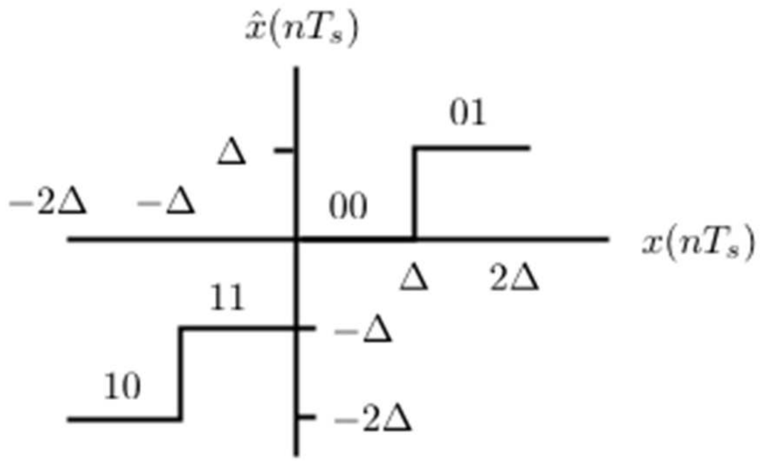
$$Y_s(\Omega) \quad = \quad X_s(\Omega)H(j\Omega) = \left[\frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)\right]\frac{\sin(\Delta\Omega/2)}{\Omega/2}e^{-j\Omega\Delta/2}$$

spectrum of ideally sampled signal

weight due to zero-order hold system

Quantization and Coding

- Quantizer: amplitude discretization of the the sampled signal $x_s(t)$
- Coder: distinct binary code for each level of quantizer



Consider

$$x(nT_s) = x(t)|_{t=nT_S}$$

Input $x(nT_s)$, output $\hat{x}(nT_s)$

4-level quantizer: $k\Delta \le x(nT_s) < (k+1)\Delta \quad \Rightarrow \quad \hat{x}(nT_s) = k\Delta \qquad k = -2, -1, 0, 1$

Quantization

$$\begin{aligned} -2\Delta \le x(nT_s) < -\Delta \quad &\Rightarrow \quad \hat{x}(nT_s) = -2\Delta \\ -\Delta \le x(nT_s) < 0 \quad &\Rightarrow \quad \hat{x}(nT_s) = -\Delta \\ 0 \le x(nT_s) < \Delta \quad &\Rightarrow \quad \hat{x}(nT_s) = 0 \\ \Delta \le x(nT_s) < 2\Delta \quad &\Rightarrow \quad \hat{x}(nT_s) = \Delta \end{aligned}$$

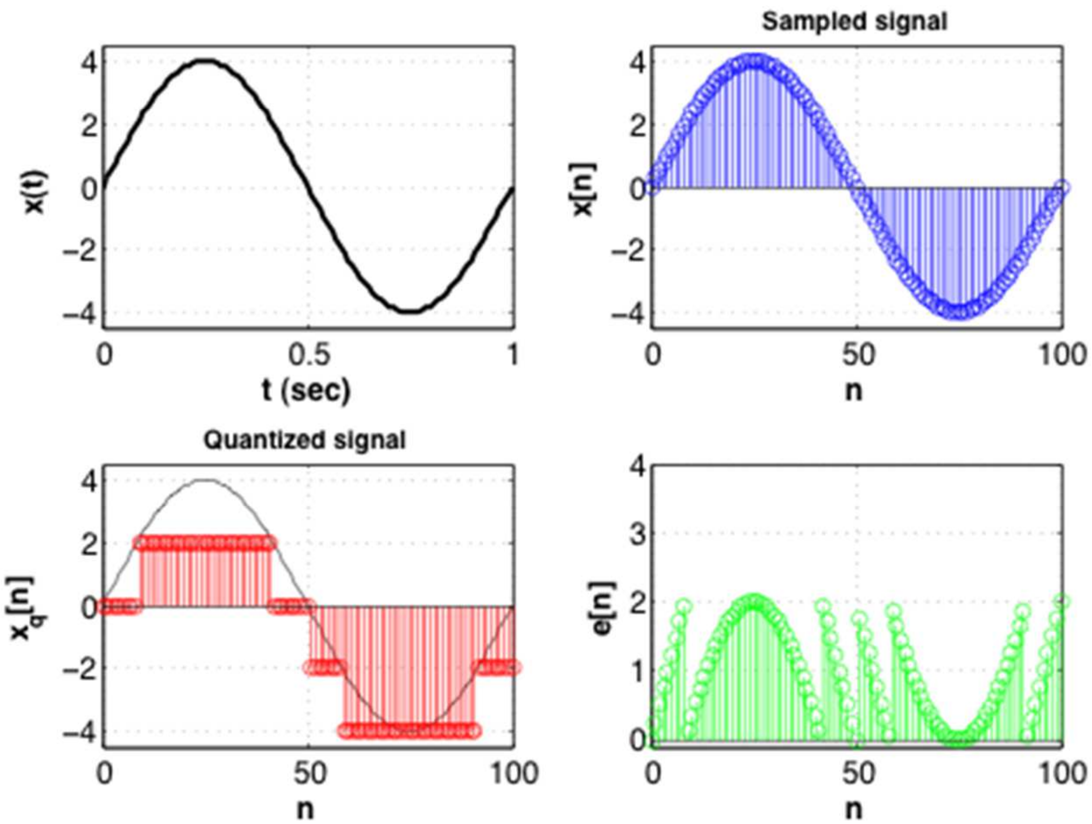
Coding

$$\begin{aligned} \hat{x}(nT_s) = -2\Delta \quad &\Rightarrow \quad 10 \\ \hat{x}(nT_s) = -\Delta \quad &\Rightarrow \quad 11 \\ \hat{x}(nT_s) = 0\Delta \quad &\Rightarrow \quad 00 \\ \hat{x}(nT_s) = \Delta \quad &\Rightarrow \quad 01 \end{aligned}$$

Quantization error $\varepsilon(nT_s) = x(nT_s) - \hat{x}(nT_s)$

$\hat{x}(nT_s) \le x(nT_s) \le \hat{x}(nT_s) + \Delta$ subtracting $\hat{x}(nT_s) \quad \Rightarrow \quad 0 \le \varepsilon(nT_s)$

To decrease $\varepsilon(nT_s)$ reduce quantization step Δ or increase number of bits


Sampled signal
Quantized signal
x(t)
t (sec)
x[n]
n
$x_q[n]$
n
e[n]
n

What have we accomplished?

- How to convert an analog signal into discrete-time and digital signal
- Frequency characteristics and sampling
- Reconstruction of analog signals from sampled signals
- Zero-order hold sampling and quantization

Where do we go from here?

- Theory of discrete-time signals and systems
- Z-transform and connection with Laplace
- Discrete-time Fourier analysis
- Application to control and communications