Signals and Systems Using MATLAB

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Chapter 6, 7 - Application to Control and Communications

What is in this chapter?

- Cascade, parallel and feedback connections of LTI systems
- Application of Laplace transform to classical control
- Application of Fourier transform to communications
- Introduction to analog filter design

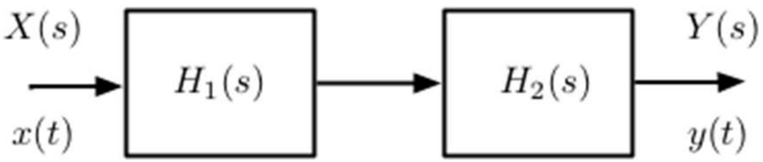
- **Classical control**: change the dynamics of a given system to be able to achieve a desired response by frequency domain methods
  - feedback connection
  - controller
  - plant (e.g. motor, chemical plant or an automobile)
  - Laplace analysis, transfer function, stability, transient analysis
- **Communications**: transmit a message over a channel to a receiver
  - transmitter
  - channel (e.g.,airwaves, telephone line)
  - receiver
  - Fourier analysis, steady state, modulation, filtering, bandwidth, spectrum
- **Filter design**: obtain a LTI system that satisfies frequency response specifications to get rid of undesirable signal component(s)
  - Filtering specifications
  - Design and implementation
  - Fourier and Laplace analyses, circuit implementation, stability, causality

System connections and block diagrams

Cascade

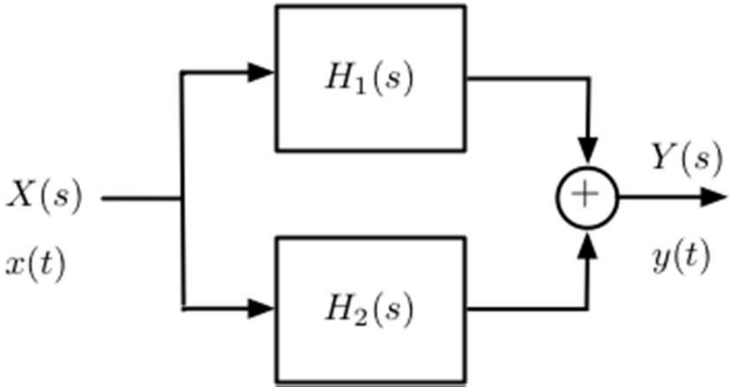
$$H(s) = H_1(s)H_2(s).$$

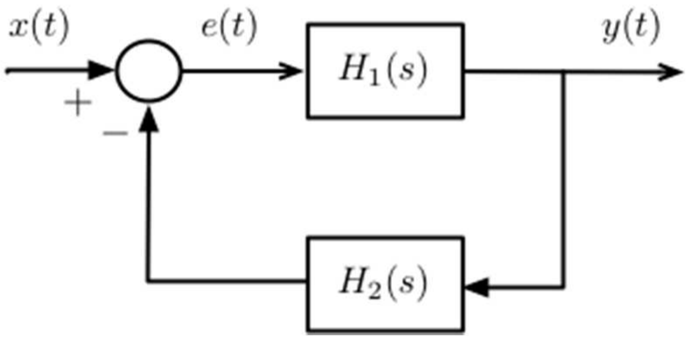
two systems are isolated



Parallel

$$H(s) = H_1(s) + H_2(s)$$



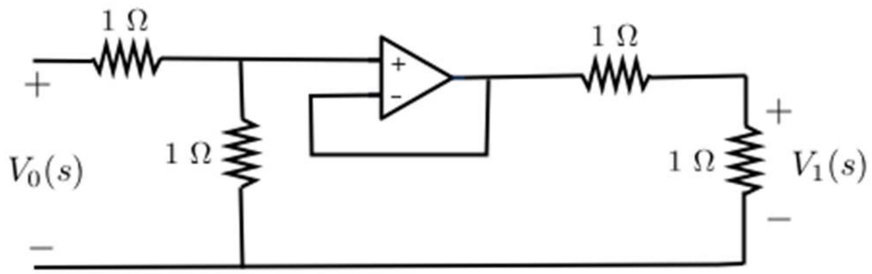


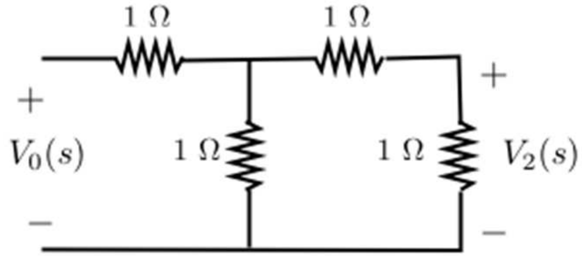
Feedback

$$H(s) = \frac{H_1(s)}{1 + H_2(s)H_1(s)}$$

- **Open-loop** transfer function: $H_{o\ell}(s) = H_1(s)$.
- **Closed-loop** transfer function: $H_{c\ell}(s) = H(s)$.

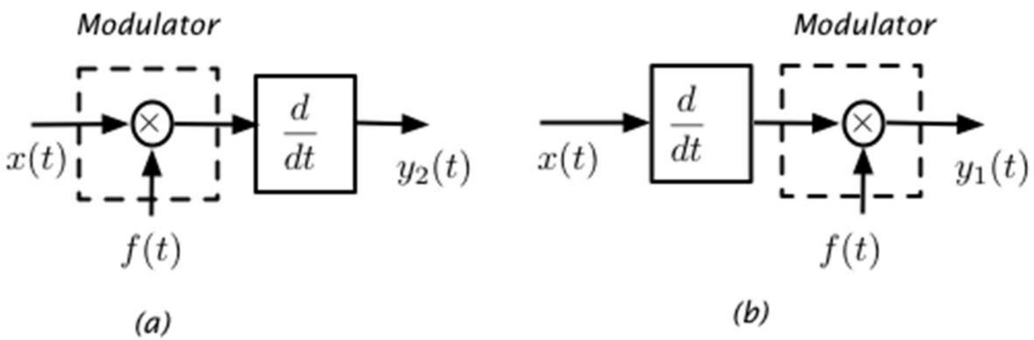
Loading in cascading





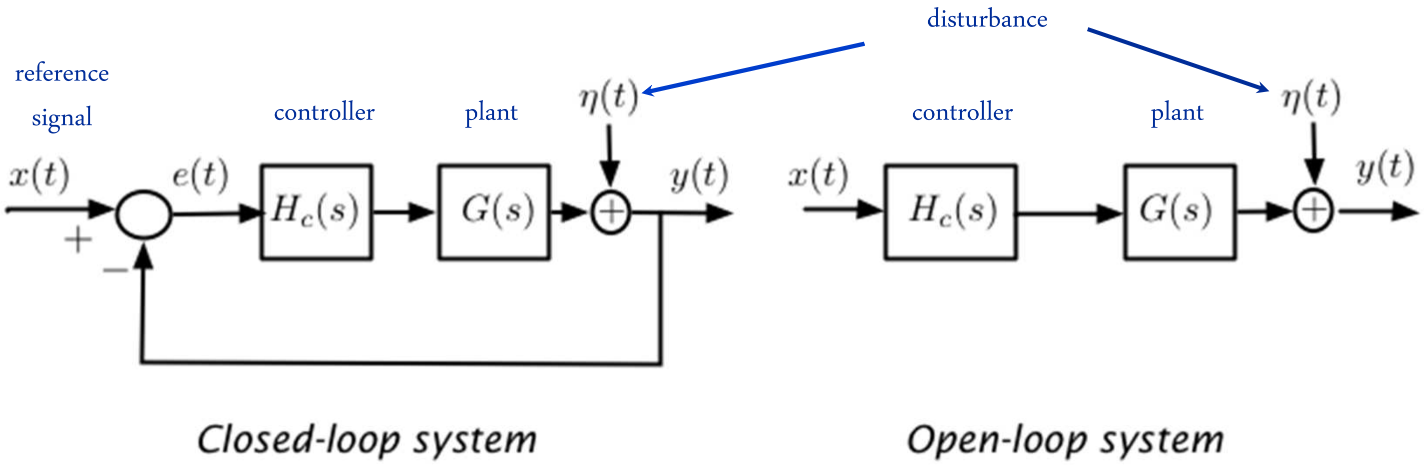
Cascading of two voltage dividers: using a voltage follower gives $V_1(s)/V_0(s) = (1/2)(1/2)$ *(top) with no loading effect; using no voltage follower* $V_2(s)/V_0(s) = 1/5 \neq V_1(s)/V_0(s)$ *due to loading.*

LTI and LTV cascading



The outputs are different, $y_1(t) \neq y_2(t)$.

Application to Classical Control



Open-loop Control: Make $y(t)$ follow $x(t)$ by minimizing error signal

$$e(t) = y(t) - x(t)$$

no disturbance, $\eta(t) = 0$,

$Y(s) = H_c(s)G(s)X(s)$ then $E(s) = Y(s) - X(s) = [H_c(s)G(s) - 1]X(s)$

to get $y(t) = x(t)$ requires $H_c(s) = 1/G(s)$ (inverse of the plant)

disturbance signal $\eta(t) \neq 0$

$Y(s) = H_c(s)G(s)X(s) + \eta(s)$ then $E(s) = [H_c(s)G(s) - 1]X(s) + \eta(s)$

$H_c(s) = 1/G(s)$ still minimizes $e(t)$ but it cannot be made zero

Closed-loop control using negative feedback:

No disturbance ($\eta(t) = 0$)

$$E(s) \quad = \quad X(s) - Y(s) \qquad Y(s) = H_c(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{X(s)}{1 + G(s)H_c(s)}$$

If $e(t) \to 0$ in steady state, i.e., $y(t)$ **tracks** $x(t)$, the poles of $E(s)$ should be in the open left–hand s-plane

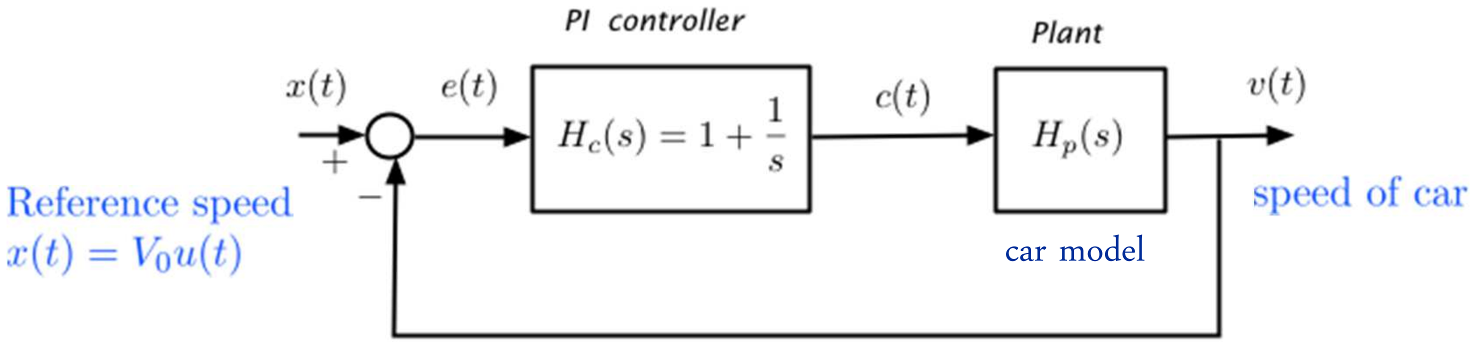
Disturbance $\eta(t) \neq 0$

$$E(s) = X(s) - Y(s) \qquad Y(s) = H_c(s)G(s)E(s) + \eta(s) \quad \Rightarrow \quad E(s) = \underbrace{\frac{X(s)}{1 + G(s)H_c(s)}}_{E_1(s)} + \underbrace{\frac{-\eta(s)}{1 + G(s)H_c(s)}}_{E_2(s)}$$

For $e(t) \to 0$ in steady state, then poles of $E_1(s)$ and $E_2(s)$ should be in the open left–hand s-plane

Closed-loop control offers more flexibility in achieving this by minimizing the effects of the disturbance

Example — A Cruise Control



Model of car in motion

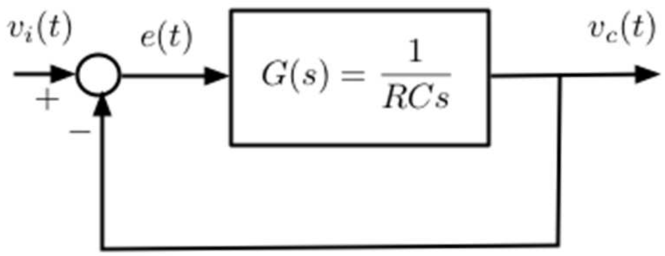
$$H_p(s) = \beta/(s+\alpha), \qquad \text{mass } \beta = 1, \text{ friction } \alpha = 1$$

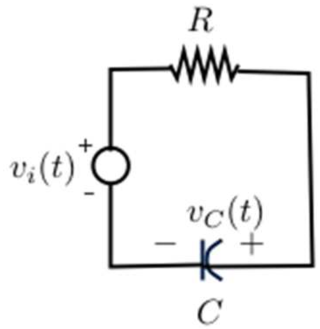
$$\begin{aligned} V(s) &= \frac{H_c(s)H_p(s)}{1+H_c(s)H_p(s)}X(s) = \frac{V_0}{s(s+1)} \\ &= \frac{B}{s+1} + \frac{V_0}{s} \end{aligned}$$

Steady-state response

$$\lim_{t\to\infty} v(t) = V_0 \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = (x(t) - v(t)) \to 0$$

i.e., speed of car is V_0





First-order System: RC circuit

$$\text{input} \quad v_i(t) = u(t) \quad \text{output} \quad v_c(t)$$

$$H(s) = \frac{V_c(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

As negative feedback system

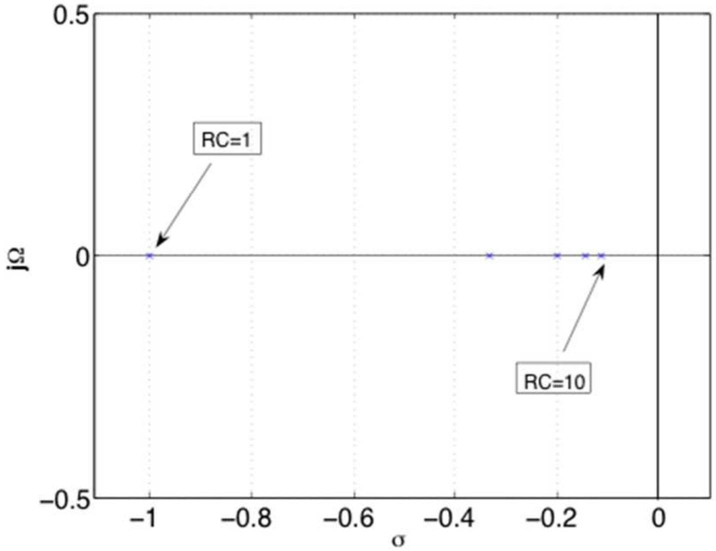
$$E(s) = V_i(s) - V_c(s)$$
$$V_c(s) = E(s)G(s)$$
$$\frac{V_c(s)}{V_i(s)} = \frac{G(s)}{1 + G(s)} = \frac{1}{1 + 1/G(s)}$$

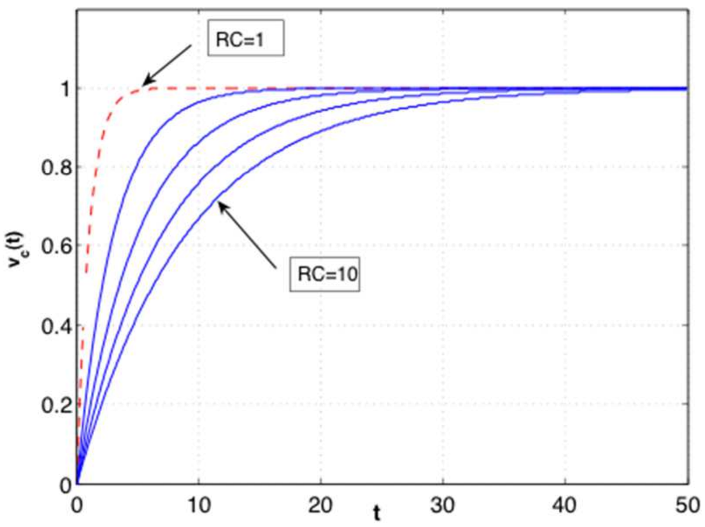
so that

$$G(s) = \frac{1}{RCs}$$

Output

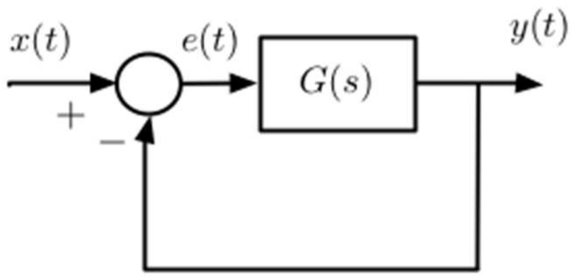
$$V_c(s) = \frac{1}{s(sRC + 1)} = \frac{1/RC}{s(s + 1/RC)} = \frac{1}{s} - \frac{1}{s + 1/RC} \quad \rightarrow \quad v_c(t) = (1 - e^{-t/RC})u(t)$$

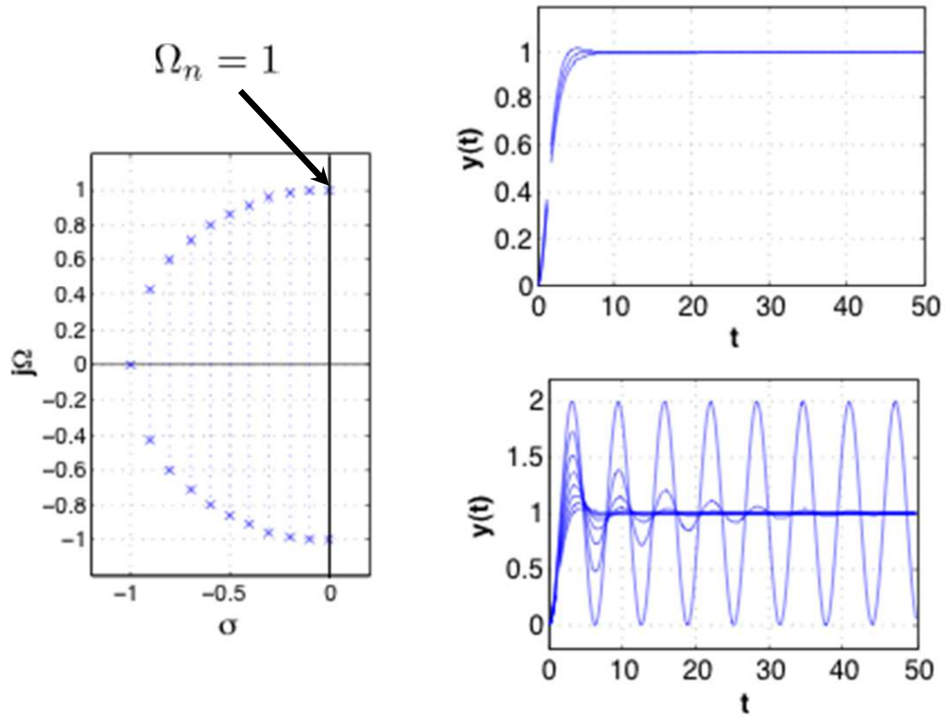




Second-order system: RLC in series

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)} = \frac{\Omega_n^2}{s^2 + 2\psi\Omega_n s + \Omega_n^2}.$$





Clustering of poles (left) and time responses of second-order feedback system for $\sqrt{2}/2 \leq \psi \leq 1$ *(top right) and* $0 \leq \psi \leq \sqrt{2}/2$ *(bottom right).*

Spectral Representation — Unification of the spectral representation of both periodic and aperiodic signals

Signal Modulation

Frequency shift: *If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair*

$$x(t)e^{j\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega - \Omega_0)$$

Modulation: *The Fourier transform of the* **modulated signal**

$$x(t)\cos(\Omega_0 t)$$

is given by

$$0.5\,[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

i.e., $X(\Omega)$ is shifted to frequencies Ω_0 and $-\Omega_0$, and multiplied by 0.5.

Remarks

- Amplitude modulation: consists in multiplying message $x(t)$ by a sinusoid of frequency higher than the maximum frequency of the incoming signal

$$\text{Modulated signal} \;\; x(t)\cos(\Omega_0 t) = 0.5[x(t)e^{j\Omega_0 t} + x(t)e^{-j\Omega_0 t}]$$
$$\mathcal{F}[x(t)\cos(\Omega_0 t)] = 0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_o)]$$

  Modulation shifts the frequencies of $x(t)$ to frequencies around $\pm\Omega_0$

- Modulation using a sine, instead of a cosine, changes the phase of the Fourier transform of the incoming signal besides performing the frequency shift

- According to eigenfunction property of LTI systems, modulation systems are not LTI.

Example Modulate a carrier $\cos(10t)$ with:

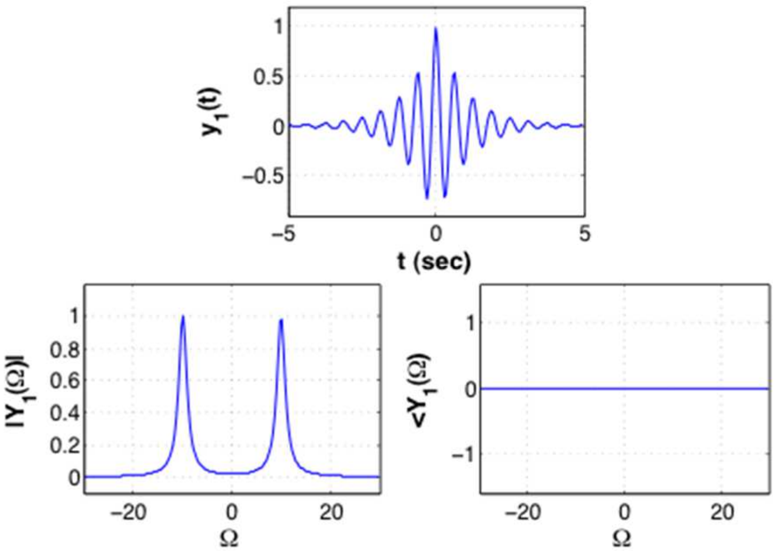
1. $x_1(t) = e^{-|t|}, \; -\infty < t < \infty.$ $x_1(t)$ is low-pass signal see spectrum before

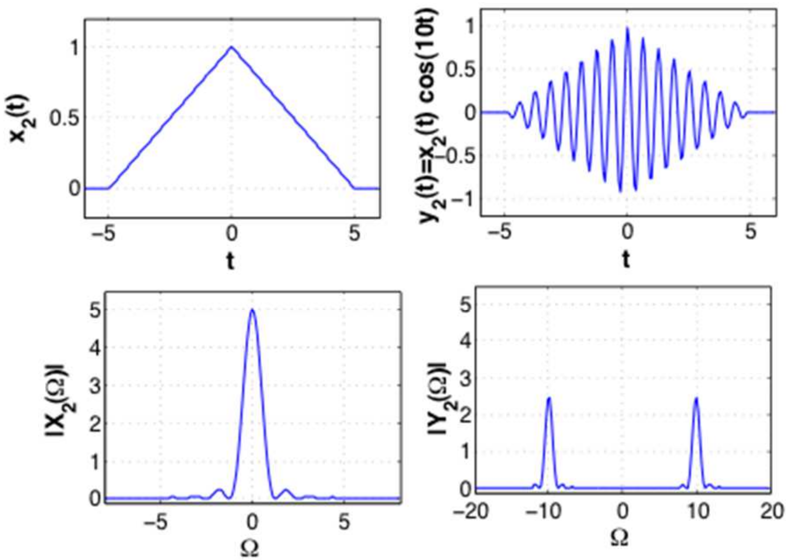
2. $x_2(t) = 0.2[r(t+5) - 2r(t) + r(t+5)].$

The modulated signals are

$$(i) \quad y_1(t) = x_1(t)\cos(10t) = e^{-|t|}\cos(10t), \qquad -\infty < t < \infty$$

$$(ii) \quad y_2(t) = x_2(t)\cos(10t) = 0.2[r(t+5) - 2r(t) + r(t+5)]\cos(10t)$$





Why Modulation? Modulation changes frequency content of a message from its baseband frequencies to higher frequencies making its transmission over the airwaves possible

Music ($0 \leq f \leq 22$KHz), and speech ($100 \leq f \leq 5$KHz) relatively low frequency signals requiring an antenna of length

$$\frac{\lambda}{4} = \frac{3 \times 10^8}{4f} \quad \text{meters}$$

$$\text{if } f = 30\text{KHz} \Rightarrow \text{ length of antenna } 2.5\text{km} \approx 1.5\text{miles}$$

thus need to increase baseband frequencies.

Fourier Transform of Periodic Signals

A periodic signal $x(t)$ of period T_0:

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \qquad \Leftrightarrow \qquad X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

Fourier series of $x(t)$:

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \qquad \Omega_0 = 2\pi/T_0$$

$$X(\Omega) = \sum_k \mathcal{F}[X_k e^{jk\Omega_0 t}] = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

Remarks

- $|X(\Omega)|$ vs Ω, the Fourier magnitude spectrum of periodic $x(t)$ is analogous to its line spectrum
- Direct computation

$$\begin{aligned}
\mathcal{F}[\cos(\Omega_0 t)] &= \mathcal{F}[0.5e^{j\Omega_0 t} + 0.5e^{-j\Omega_0 t}] = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0) \\
\mathcal{F}[\sin(\Omega_0 t)] &= \mathcal{F}\left[\frac{0.5}{j}e^{j\Omega_0 t} - \frac{0.5}{j}e^{-j\Omega_0 t}\right] = \frac{\pi}{j}\delta(\Omega - \Omega_0) - \frac{\pi}{j}\delta(\Omega + \Omega_0) \\
&= \pi e^{-j\pi/2}\delta(\Omega - \Omega_0) + \pi e^{j\pi/2}\delta(\Omega + \Omega_0)
\end{aligned}$$

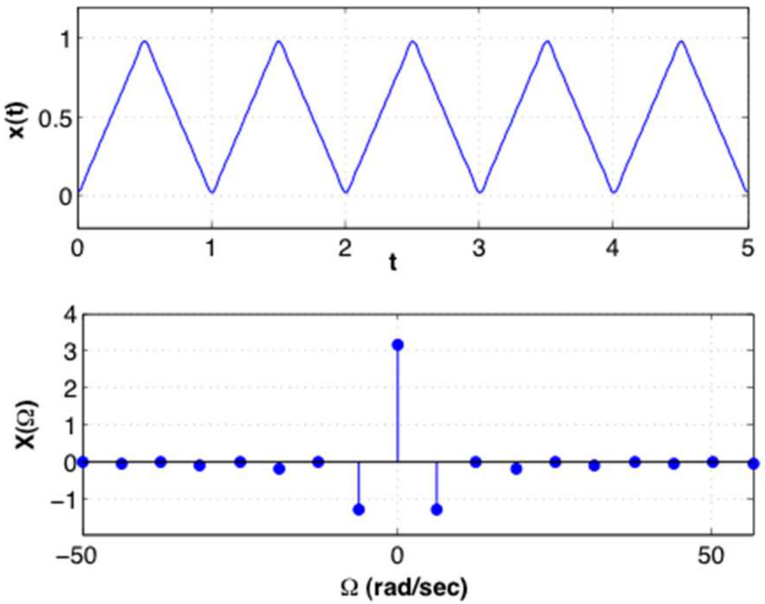
Example Triangular pulses $x(t)$ with a period

$$x_1(t) = r(t) - 2r(t-0.5) + r(t-1),\ \Omega_0 = 2\pi$$

find $X(\Omega)$

$$\begin{aligned} X_1(s) &= \frac{1}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right) = \frac{e^{-0.5s}}{s^2}\left(e^{0.5s} - 2 + e^{-0.5s}\right) \\ \text{FS coefficients}\ \ X_k &= \frac{1}{T_0}X_1(s)|_{s=j2\pi k} = \frac{1}{(j2\pi k)^2}2(\cos(\pi k)-1)e^{-j\pi k} \\ &= (-1)^{(k+1)}\frac{\cos(\pi k)-1}{2\pi^2 k^2} = (-1)^k\frac{\sin^2(\pi k/2)}{\pi^2 k^2} \\ X(0) &= 0.5 \end{aligned}$$

$$\text{FT}:\quad X(\Omega) = 2\pi X_0\delta(\Omega) + \sum_{k=-\infty,\neq 0}^{\infty} 2\pi X_k \delta(\Omega - 2k\pi)$$



Parseval's Energy Conservation *A finite-energy signal $x(t)$, with Fourier transform $X(\Omega)$, its energy is*

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

$|X(\Omega)|^2$ *energy density : energy per frequency*
$|X(\Omega)|^2$ *vs* Ω*: energy spectrum of $x(t)$, energy of the signal distributed over frequency*

Example Is $\delta(t)$ a finite energy signal?
Using Parseval's result: $\mathcal{F}\delta(t) = 1$ for all frequencies then its energy is infinite
In time-domain:

$$p_\Delta(t) = \frac{1}{\Delta}[u(t+\Delta/2) - u(t-\Delta/2)] \quad \Rightarrow \quad \delta(t) \text{as } \Delta \to 0, \text{ unit area}$$
$$p^2_\Delta(t) = \frac{1}{\Delta^2}[u(t+\Delta/2) - u(t-\Delta/2)] \quad \Rightarrow \quad \delta^2(t) \text{as } \Delta \to 0, \text{ infinite area } 1/\Delta$$

$\delta(t)$ is not finite energy

Symmetry of Spectral Representations
If $X(\Omega)$ is FT of real-valued signal $x(t)$, periodic or aperiodic then

- *Magnitude $|X(\Omega)|$ even function of Ω:*

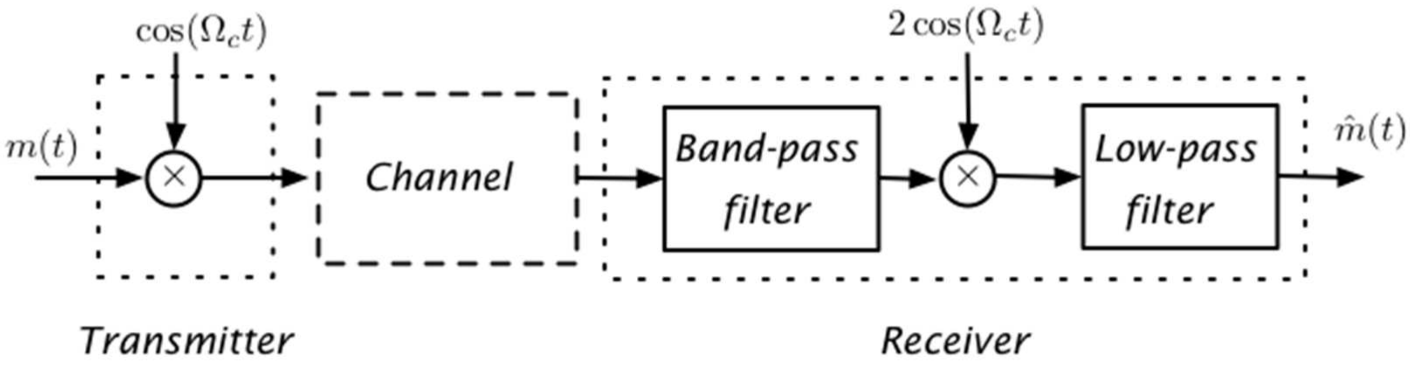
$$|X(\Omega)| = |X(-\Omega)|$$

- *Phase $\angle X(\Omega)$ odd function of Ω:*

$$\angle X(\Omega) = -\angle X(-\Omega)$$

$\|X(\Omega)\|$ *vs* Ω	**Magnitude Spectrum**
$\angle X(\Omega)$ *vs* Ω	**Phase Spectrum**
$\|X(\Omega)\|^2$ *vs* Ω	**Energy/Power Spectrum.**

AM Suppressed Carrier (AM-SC)



Transmitter

$$s(t) = m(t)\cos(\Omega_c t) \quad \Omega_c >> 2\pi f_0,\ f_0 \text{max frequency in } \ m(t)$$

transform of $s(t)$

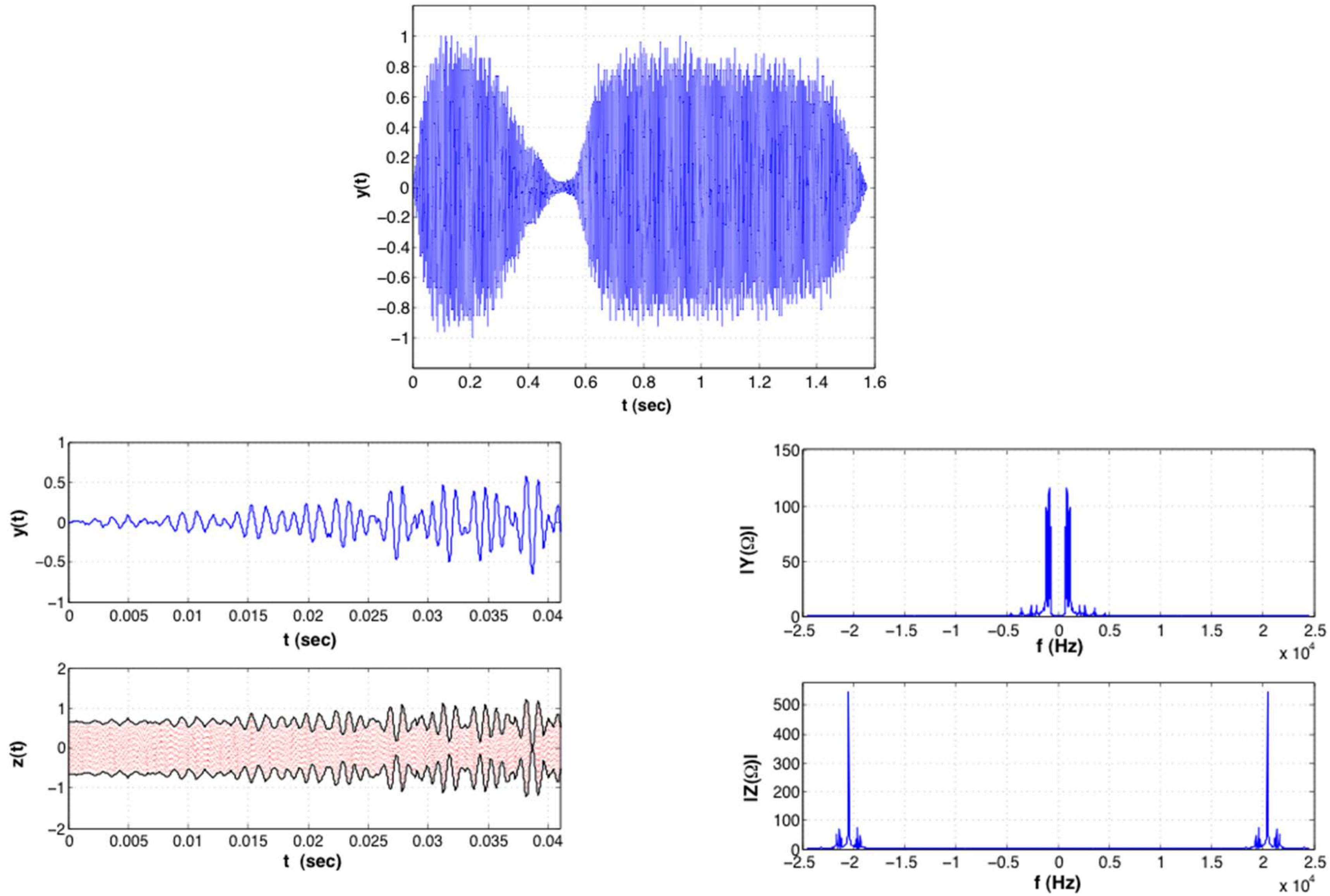
$$S(\Omega) = \frac{1}{2}\left[M(\Omega - \Omega_c) + M(\Omega + \Omega_c)\right] \quad M(\Omega) \ \text{ spectrum of } \ m(t)$$

Receiver

$$r(t) = 2s(t)\cos(\Omega_c t)$$

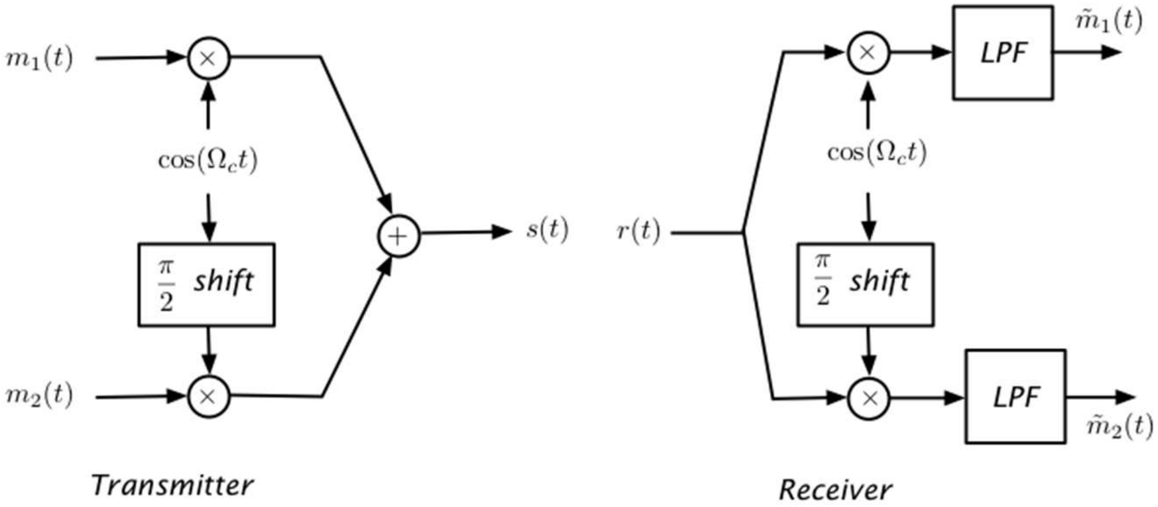
$$R(\Omega) = S(\Omega - \Omega_c) + S(\Omega + \Omega_c) = M(\Omega) + \frac{1}{2}\left[M(\Omega - 2\Omega_c) + M(\Omega + 2\Omega_c)\right]$$

$M(\Omega)$ and $m(t)$ obtained passing $r(t)$ through LPF



Commercial AM modulation: original signal (top), part of original signal and corresponding AM modulated signal (bottom left), spectrum of the original signal and of the modulated signal (bottom right).

Quadrature Amplitude Modulation (QAM)

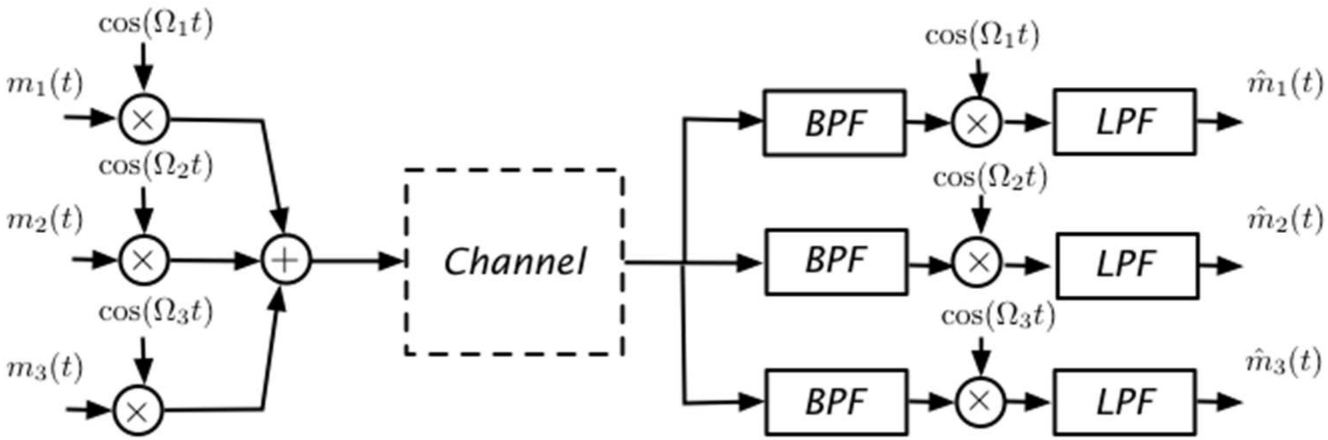


QAM: conserving bandwidth

$$s(t) = m_1(t)\cos(\Omega_c t) + m_2(t)\sin(\Omega_c t) \quad \text{messages} \quad m_1(t),\ m_2(t)$$

Frequency-division Multiplexing

Optimizing use of spectrum



Phase Modulation: $s_{PM}(t) = \cos(\Omega_c t + K_f m(t))$

Frequency Modulation: $s_{FM}(t) = \cos(\Omega_c t + \Delta\Omega \int_{-\infty}^{t} m(\tau)d\tau)$

FM modulation paradox

In *amplitude* modulation the bandwidth depends on the *frequency* of the message, while in *frequency* modulation the bandwidth depends on the *amplitude* of the message. (E. Craig)

Example — FM Simulation

Narrow-band FM

$$\begin{aligned}
&\text{message} \quad m(t) = 80\sin(20\pi t)u(t) \\
&\text{carrier} \quad \cos(2\pi f_c t), \;\; f_c = 100\text{Hz} \\
&\text{FM signal} \quad x(t) = \cos(2\pi f_c t + 0.1\pi \int_{-\infty}^{t} m(\tau)d\tau)
\end{aligned}$$

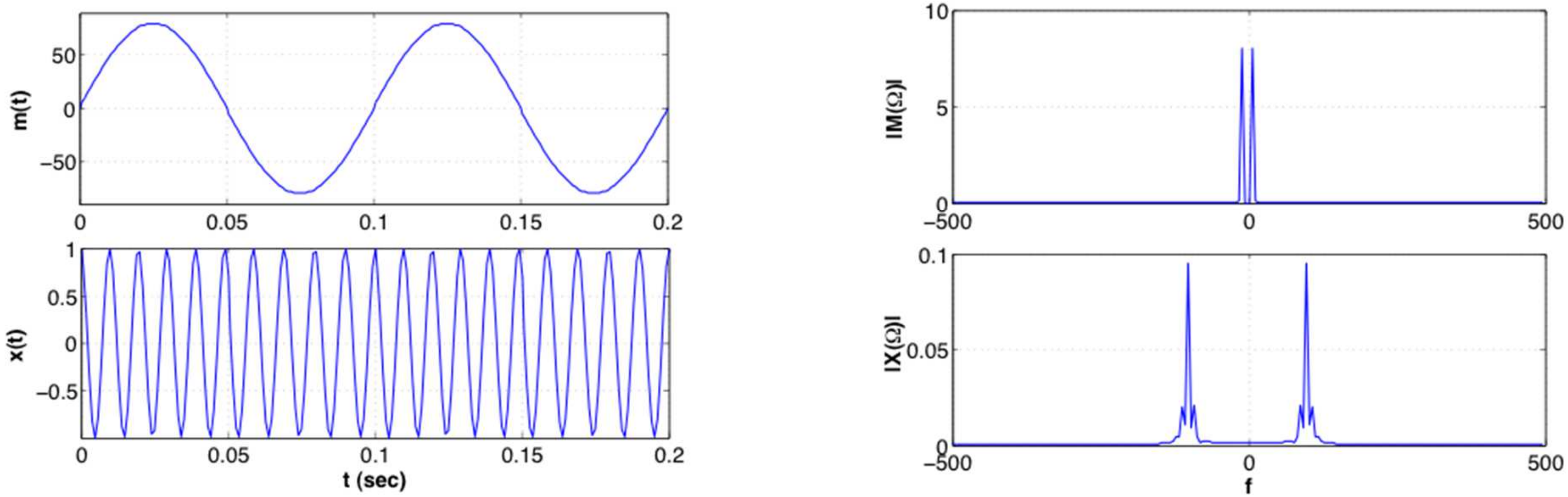
Instantaneous frequency (the derivative of the argument of the cosine) is

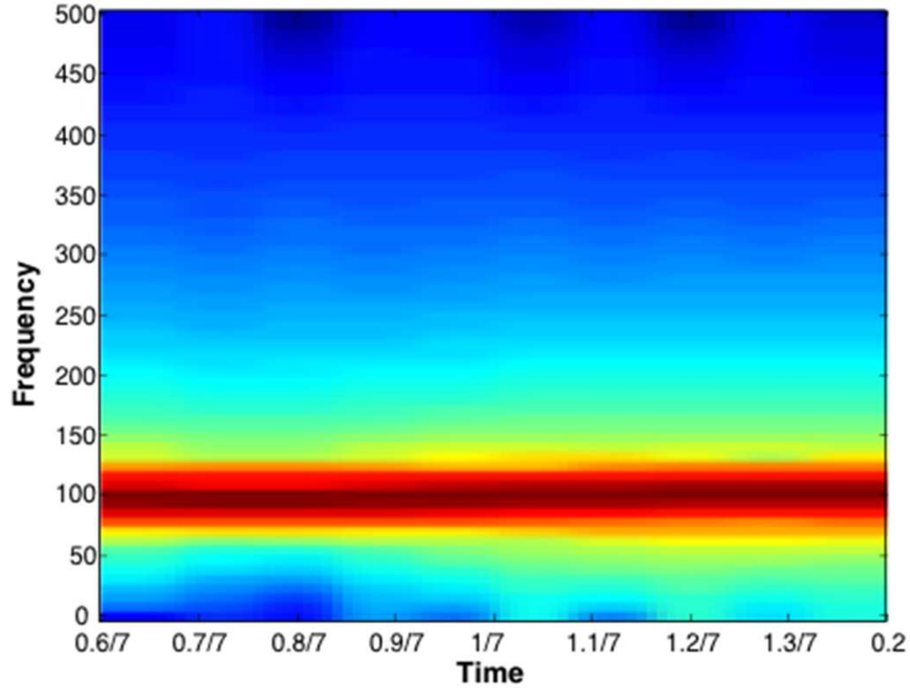
$$IF(t) = 2\pi f_c + 0.1\pi m(t) = 200\pi + 8\pi\sin(20\pi t) \approx 200\pi$$

i.e., it remains almost constant for all time

Spectrogram: Fourier transform as the signal evolves with time

Narrow-band FM



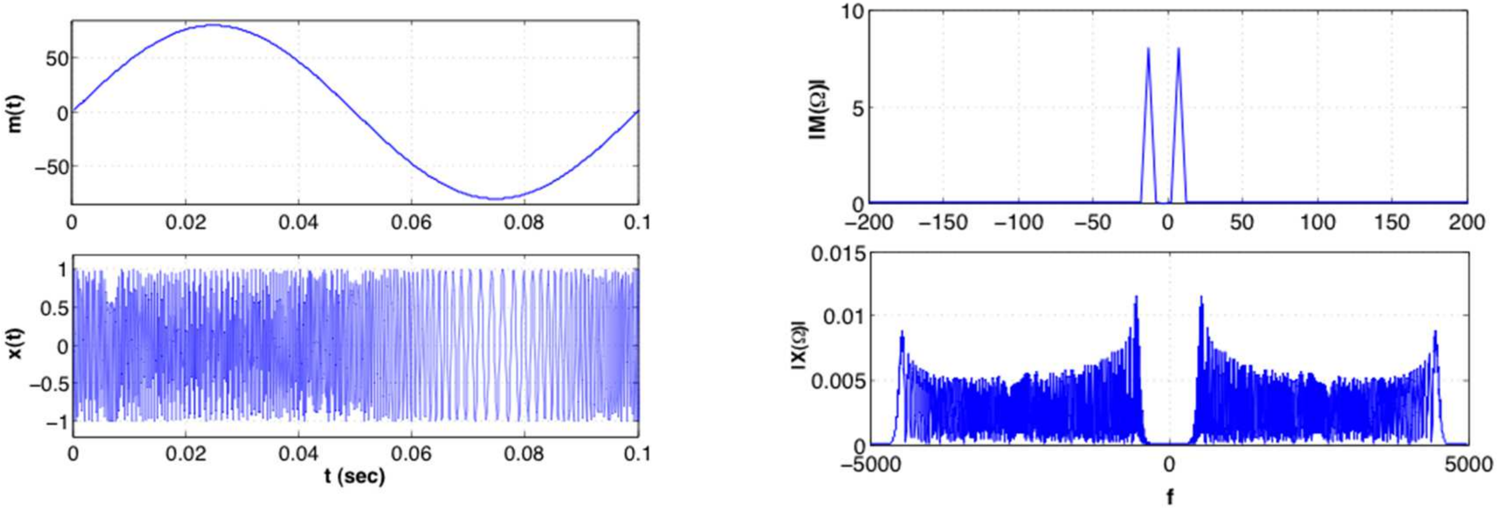


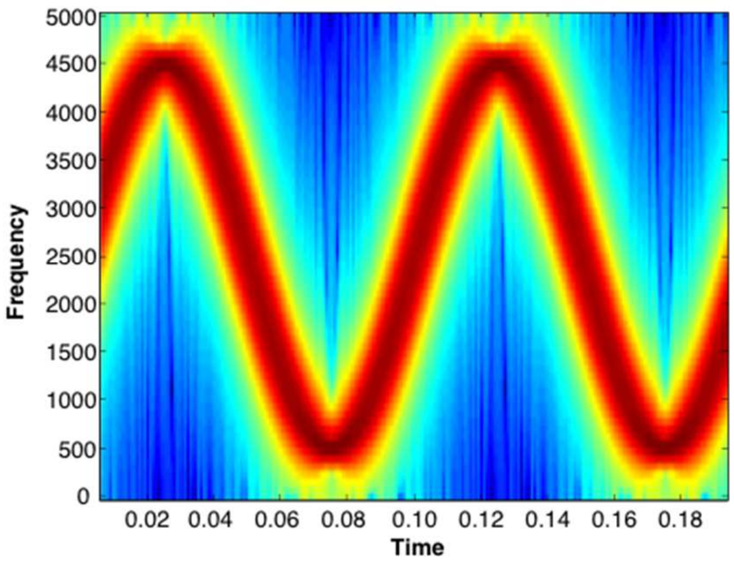
Wide-band FM

$$m_1(t) = 80\sin(20\pi t)u(t)$$

FM signal $x_i(t) = \cos(2\pi 2500t + 50\pi \int_{-\infty}^{t} m_1(\tau)d\tau)$

IF $IF_1(t) = 2\pi f_{ci} + 50\pi m_1(t)$





Convolution and Filtering *If $x(t)$ (periodic or aperiodic) is input to a stable LTI system with a frequency response $H(j\Omega) = \mathcal{F}[h(t)]$, $h(t)$ impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform*

$$Y(\Omega) = \mathcal{F}[(x * h)] = X(\Omega)\ H(j\Omega)$$

If $x(t)$ is periodic the output is also periodic with Fourier transform

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\ X_k\ H(jk\ \Omega_0)\delta(\Omega - k\Omega_0)$$

where X_k are the Fourier series coefficients of $x(t)$ and Ω_0 its fundamental frequency.

Eigenfunction property of LTI systems:
Aperiodic signals

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \Rightarrow \\ y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [X(\Omega)H(j\Omega)]\, e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\Omega)e^{j\Omega t}d\Omega \end{aligned}$$

Periodic signal of perio T_0

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k\delta(\Omega - k\Omega_0) \quad \Rightarrow \quad Y(\Omega) \quad = \quad X(\Omega)H(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0)\delta(\Omega - k\Omega_0)$$

output $y(t)$ is periodic

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{Y_k}_{X_k H(jk\Omega_0)} e^{jk\Omega_0 t}$$

Basics of Filtering

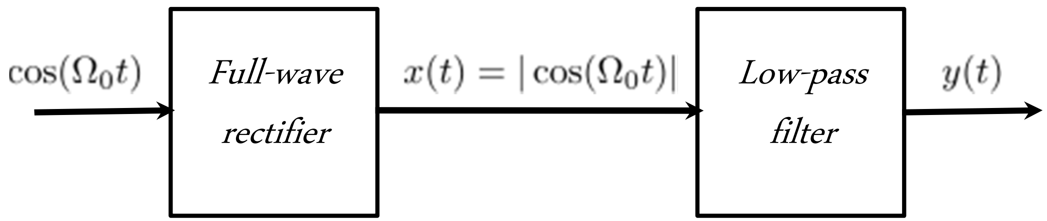
- Filtering consists in getting rid of undesirable components of a signal, e.g., noise $\eta(t)$ is added to a desired signal $x(t)$

$$y(t) = x(t) + \eta(t)$$

  Filter design: find $H(s) = B(s)/A(s)$ satisfying certain specifications to get rid of noise.

- Frequency discriminating filters keep the frequency components of a signal in a certain frequency band and attenuate the rest.

Example Obtain dc source of unity amplitude using a full-wave rectifier and a low-pass filter (it keeps only the low-frequency components)



FS coefficients:

$$X_0 = \frac{2}{\pi}$$
$$X_k = \frac{2(-1)^k}{\pi(1-4k^2)} \qquad k \neq 0$$

Filter out all harmonics and leave average component: ideal low-pass filter

$$H(j\Omega) = \begin{cases} A & -\Omega_0 < \Omega_c < \Omega_0, \quad \text{where} \quad \Omega_0 = 2\pi/T_0 = 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Ideal Filters

- Low-pass filter (keeps low-frequency components)

$$|H_{lp}(j\Omega)| = \begin{cases} 1 & -\Omega_1 \le \Omega \le \Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\angle H_{lp}(j\Omega) = -\alpha\Omega$$

- Band-pass filter (keeps middle frequency components)

$$|H_{bp}(j\Omega)| = \begin{cases} 1 & \Omega_1 \le \Omega \le \Omega_2 \quad \text{and} \quad -\Omega_2 \le \Omega \le -\Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

  linear phase in the passband

- High-pass filter (keeps high-frequency components)

$$|H_{hp}(j\Omega)| = \begin{cases} 1 & \Omega \ge \Omega_2 \quad \text{and} \quad \Omega \le -\Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

  linear phase in the passband

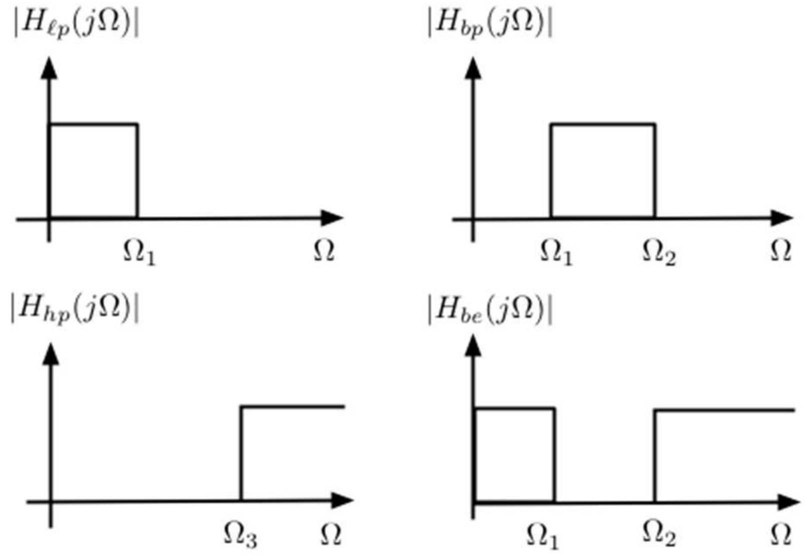
- Band-stop filter (attenuates middle frequency components)

$$|H_{bs}(j\Omega)| = 1 - |H_{bp}(j\Omega)|$$

- All-pass filter (keeps all frequency components, changes phase)

$$|H_{ap}(j\Omega)| = |H_{lp}(j\Omega)| + |H_{bp}(j\Omega)| + |H_{hp}(j\Omega)| = 1$$

- Multi-band filter: combination of the low-, band-, and high-pass filters



Example Gibb's phenomenon of Fourier series: ringing around discontinuities of periodic signals. Consider a periodic train of square pulses $x(t)$ of period T_0 displaying discontinuities at $kT_0/2$, for $k = \pm 1, \pm 2, \cdots$. Show Gibb's phenomenon is due to ideal low-pass filtering.
Ideal low-pass filter

$$H(j\Omega) = \begin{cases} 1 & -\Omega_c \leq \Omega \leq \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

Periodic signal $x(t)$, of fundamental frequency $\Omega_0 = 2\pi/T_0$, is

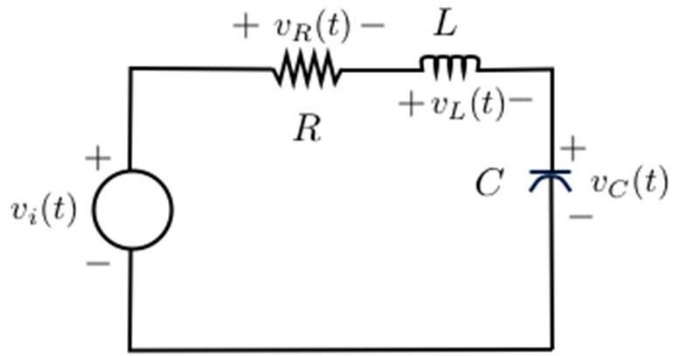
$$X(\Omega) = \mathcal{F}[x(t)] = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0)$$

Output of the filter:

$$\begin{aligned} x_N(t) &= \mathcal{F}^{-1}[X(\Omega)H(j\Omega)] = \mathcal{F}^{-1}\left[\sum_{k=-N}^{N} 2\pi X_k \delta(\Omega - k\Omega_0)\right] \\ &= [x * h](t) \qquad \text{cutoff frequency:} \quad N\Omega_0 < \Omega_c < (N+1)\Omega_0 \end{aligned}$$

convolution around the discontinuities of $x(t)$ causes ringing before and after them, independent of N

Example Obtain different filters from an RLC circuit by choosing different outputs. Let R= 1 Ω, L = 1 H, and C= 1 F, and IC=0



Low-pass Filter: Output $V_c(s)$

$$H_{lp}(s) = \frac{V_C(s)}{V_i(s)} = \frac{1}{s^2 + s + 1}$$

- input a dc source (frequency $\Omega = 0$), inductor short circuit, capacitor open circuit, so $V_c(s) = V_i(s)$
- input of very high frequency, $\Omega \to \infty$, inductor open circuit, capacitor short circuit $V_c(s) = V_i(s) = 0$

High-pass Filter: Output $V_L(s)$

$$H_{hp}(s) = \frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2 + s + 1}$$

- dc input (frequency zero), inductor is short circuit $V_L(s) = 0$
- input of very high frequency, $\Omega \to \infty$, inductor is open circuit $V_L(s) = V_i(s)$

Band-pass Filter: Output $V_R(s)$

$$H_{bp}(s) = \frac{V_R(s)}{V_i(s)} = \frac{s}{s^2 + s + 1}$$

- For zero frequency, capacitor is open circuit so voltage across the resistor is zero
- For very high frequency, inductor is open circuit, voltage across resistor is zero
- For some middle frequency, serial LC combination resonates (zero impedance) maximum voltage across resistor

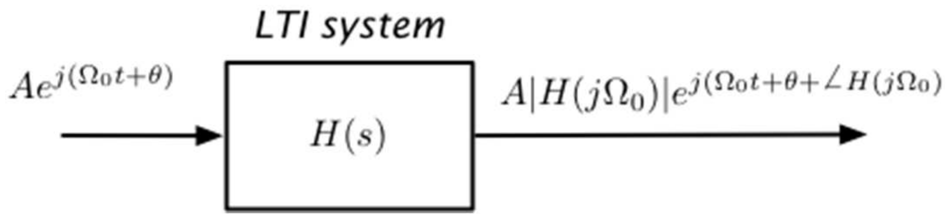
Band-stop Filter: Output voltage across inductor and the capacitor

$$H_{bs}(s) = \frac{s^2 + 1}{s^2 + s + 1}$$

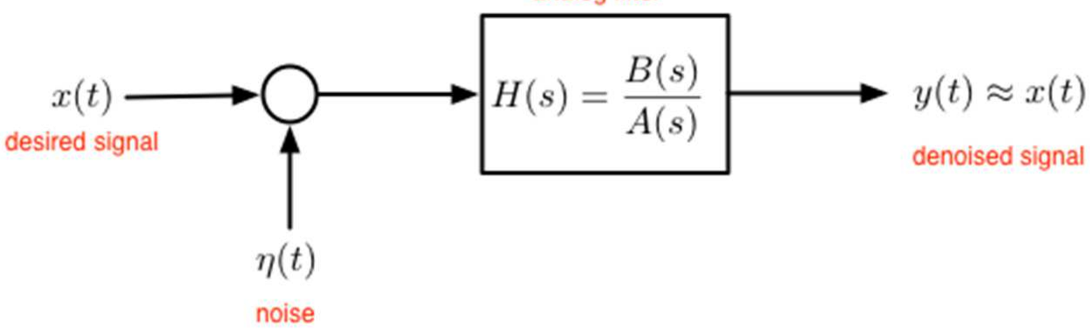
- At low and high frequencies, LC is open-circuit, $V_{LC}(s) = V_i(s)$
- At the resonance frequency $\Omega_r = 1$ the impedance of the LC connection is zero, so the output voltage is zero

Analog Filter Design

Filtering: To get rid of undesirable frequency components of a signal



Example: denoising



LPF Design

Magnitude squared function of analog LPF

$$|H(j\Omega)|^2 = \frac{1}{1 + f(\Omega^2)}$$

low frequencies $f(\Omega^2) \approx 0 \quad \Rightarrow \quad |H(j\Omega)|^2 \approx 1$
high frequencies $f(\Omega^2) \to \infty \quad \Rightarrow \quad |H(j\Omega)|^2 \to 0$ Design issues

1. selection of $f(.)$,
2. factorization to get $H(s)$ from the magnitude squared function

Example: Nth-order Butterworth LPF

$$|H_N(j\Omega)|^2 = \frac{1}{1 + \left[\frac{\Omega}{\Omega_{hp}}\right]^{2N}} \qquad \Omega_{hp}\text{half-power frequency}$$

Approximation

$$\begin{aligned} \Omega << \Omega_{hp} \quad &\Rightarrow \quad |H_N(j\Omega)| \approx 1 \\ \Omega >> \Omega_{hp} \quad &\Rightarrow \quad |H_N(j\Omega)| \to 0 \end{aligned}$$

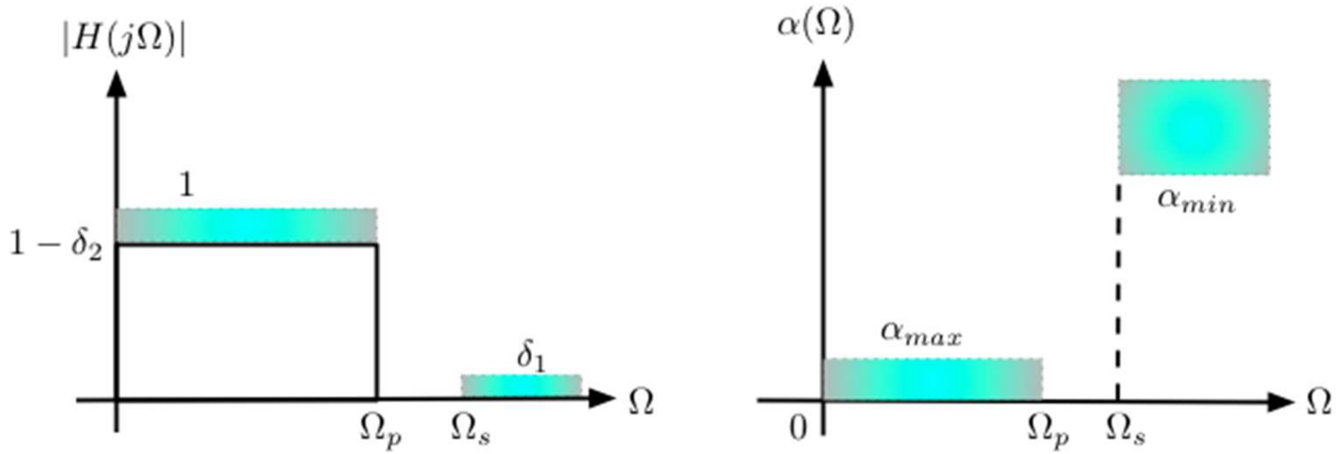
Factorization

$$\text{normalized variable} \quad S = s/\Omega_{hp} \quad \Rightarrow \quad S/j = \Omega' = \Omega/\Omega_{hp}$$

$$H(S)H(-S) = \frac{1}{1 - S^{2N}} = \underbrace{\frac{1}{D(S)}}_{H(S)} \underbrace{\frac{1}{D(-S)}}_{H(-S)}$$

Butterworth low-pass filter: $H(s) = H(S)|_{S=s/\Omega_{hp}}$

Filter Specifications --- low-pass



$$\text{Passband} \quad 1-\delta_2 \le |H(j\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$$
$$\text{Stopband} \quad 0 \le |H(j\Omega)| \le \delta_1 \qquad \Omega \ge \Omega_s, \quad \text{small} \ \ \delta_1, \ \delta_2$$

No specifications in transition region
Phase to be linear in passband

Loss function: $\alpha(\Omega) = -10\log_{10}|H(j\Omega)|^2 = -20\log_{10}|H(j\Omega)|$ dBs

Passband: $0 \le \alpha(\Omega) \le \alpha_{max} \qquad 0 \le \Omega \le \Omega_p$
Stopband: $\alpha(\Omega) \ge \alpha_{min} \qquad \Omega \ge \Omega_s$
$\alpha_{max} = -20\log_{10}(1-\delta_2)$
$\alpha_{min} = -20\log_{10}(\delta_1)$

Butterworth low-pass filter design

Magnitude squared function

$$|H_N(j\Omega')|^2 = \frac{1}{1 + (\Omega/\Omega_{hp})^{2N}} \qquad \Omega' = \frac{\Omega}{\Omega_{hp}}$$

Ω_{hp} half-power or -3dB frequency

Design

$$\alpha(\Omega) = -10\log_{10}|H_N(\Omega/\Omega_{hp})|^2 = 10\log_{10}(1 + (\Omega/\Omega_{hp})^{2N})$$

Loss specifications

$$0 \le \alpha(\Omega) \le \alpha_{max} \qquad 0 \le \Omega \le \Omega_p$$

$$\alpha_{min} \le \alpha(\Omega) < \infty \qquad \Omega \ge \Omega_s$$

Half-power frequency $\quad \dfrac{\Omega_p}{(10^{0.1\alpha_{max}} - 1)^{1/2N}} \le \Omega_{hp} \le \dfrac{\Omega_s}{(10^{0.1\alpha_{min}} - 1)^{1/2N}}$

Minimum order $\quad N \ge \dfrac{\log_{10}[(10^{0.1\alpha_{min}} - 1)/(10^{0.1\alpha_{max}} - 1)]}{2\log_{10}(\Omega_s/\Omega_p)}$

Normalized poles: for $S = s/\Omega_{hp}$

$$S_k = e^{j(2k-1+N)\pi/(2N)} \qquad k = 1, \cdots, 2N$$

Chebyshev low-pass filter design

Magnitude squared function

$$|H_N(\Omega')|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\Omega/\Omega_p)} \qquad \Omega' = \frac{\Omega}{\Omega_p}$$

N minimum order

ε ripple factor

$C_N(.)$ Chebyshev polynomials

Chebyshev polynomials

$$C_N(\Omega') = \begin{cases} \cos(N\cos^{-1}(\Omega')) & |\Omega'| \le 1 \\ \cosh(N\cosh^{-1}(\Omega')) & |\Omega'| > 1 \end{cases}$$

Recursion: $C_{N+1}(\Omega') + C_{N-1}(\Omega') = 2\cos(\theta)\cos(N\theta) = 2\Omega' C_N(\Omega')$

$C_0(\Omega') = 1$

$C_1(\Omega') = \Omega'$

Design

Loss function

$$\alpha(\Omega') = 10\log_{10}\left[1+\varepsilon^2 C_N^2(\Omega')\right] \qquad \Omega' = \frac{\Omega}{\Omega_p}$$

- Ripple factor ε:

$$\varepsilon = \sqrt{10^{0.1\alpha_{max}} - 1}$$

- Minimum order: N

$$N \geq \frac{\cosh^{-1}\left(\left[\frac{10^{0.1\alpha_{min}}-1}{10^{0.1\alpha_{max}}-1}\right]^{0.5}\right)}{\cosh^{-1}(\frac{\Omega_s}{\Omega_p})}$$

- Half-power frequency:

$$\Omega_{hp} = \Omega_p \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right]$$

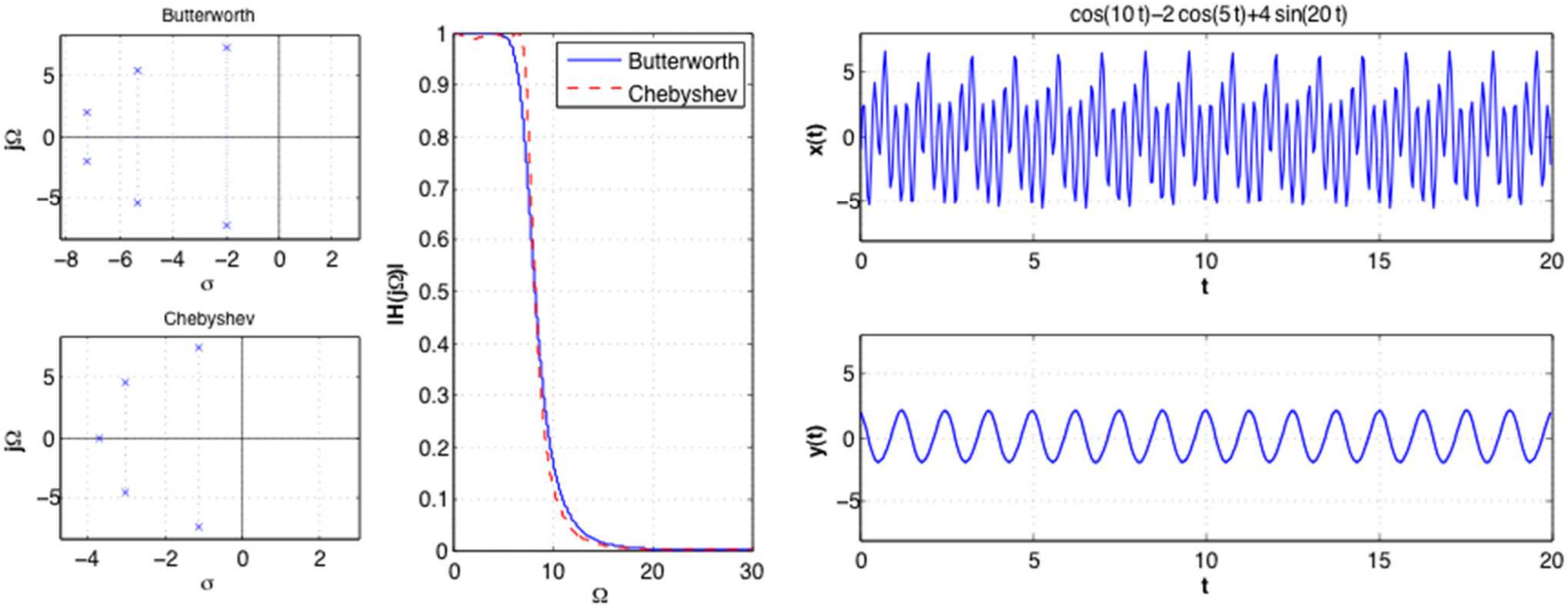
Example Compare performance of Butterworth and Chebyshev low-pass filters in filtering

$$x(t) = [-2\cos(5t) + \cos(10t) + 4\sin(20t)]u(t)$$

Two filters must have same half-power frequency
Specifications

$$\alpha_{max} = 0.1 \;\; \text{dB}, \;\; \Omega_p = 5 \;\; \text{rad/sec}$$
$$\alpha_{min} = 15 \;\; \text{dB}, \;\; \Omega_s = 10 \;\; \text{rad/sec}$$

dc loss of 0 dB



Frequency transformations

Lowpass-Lowpass $\quad S = \dfrac{s}{\Omega_0}$

Lowpass-Highpass $\quad S = \dfrac{\Omega_0}{s}$

Lowpass-Bandpass $\quad S = \dfrac{s^2 + \Omega_0}{s\ BW}$

Lowpass-Band-eliminating $\quad S = \dfrac{s\ BW}{s^2 + \Omega_0}$

S normalized $\ s$ final variable

Ω_0 desired cut-off frequency and $\ BW\ $ desired bandwidth

Example

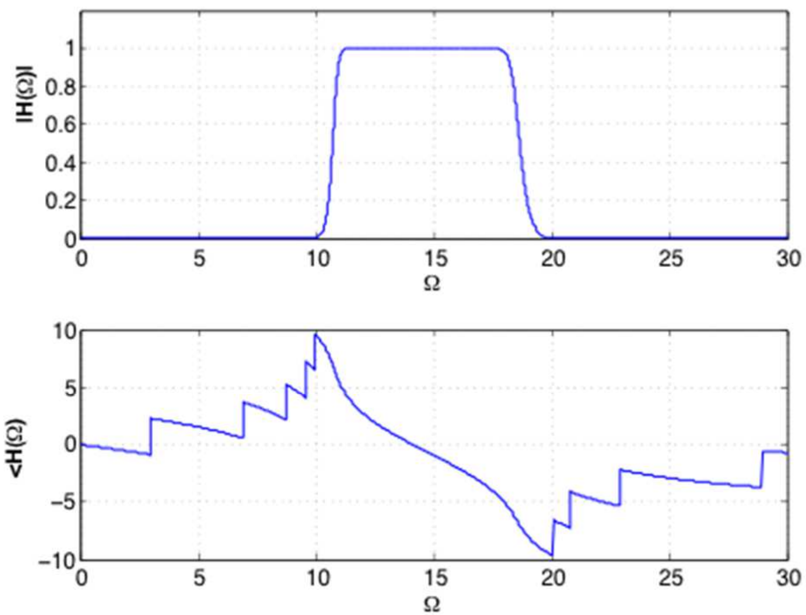
1. Use *cheby2* to design bandpass filter with specifications

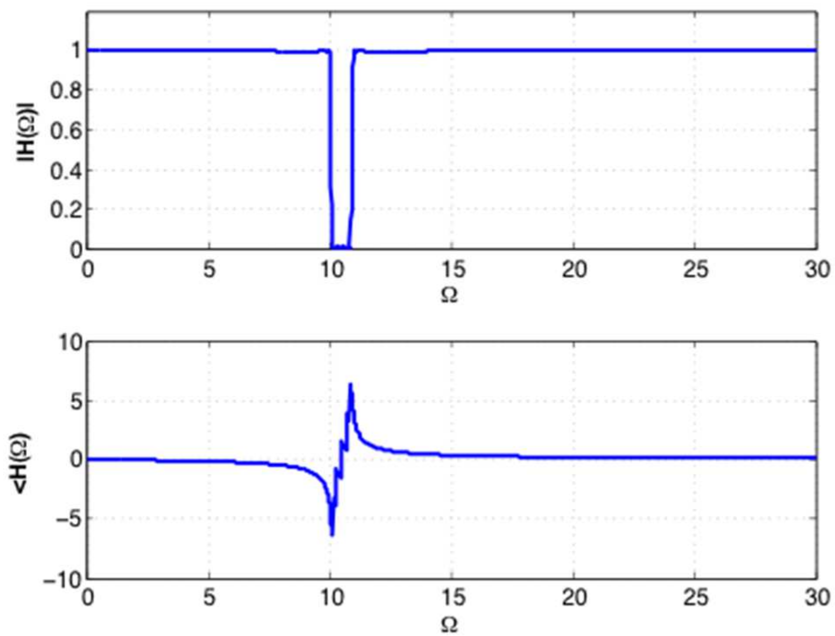
$$\text{order} \quad N = 10$$
$$\alpha(\Omega) = 60 \quad \text{dB in the stopband}$$
$$\text{passband frequencies} \ [10, \quad 20] \ \text{rad/sec}$$

   unit gain in the passband

2. Use *ellip* to design bandstop filter with unit gain in the passbands and the following specifications

$$\text{order} \quad N = 20$$
$$\alpha(\Omega) = 0.1 \quad \text{dB in the passband}$$
$$\alpha(\Omega) = 40 \quad \text{dB in the stopband}$$
$$\text{passband frequencies} \ [10, \quad 11] \ \text{rad/sec}$$





What have we accomplished?

- Illustrated application of Laplace analysis to classical control
- Showed application of Fourier steady-state analysis to communications
- Introduced the design of analog filters

Where do we go from here?

- Discrete-time signals and systems
- Application of Fourier analysis to sampling
- Transform methods and connection with Laplace
- Application to control and communications
- Design of discrete filters