Signals and Systems Using MATLAB

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Chapter 5 - Frequency Analysis

The Fourier Transform

What is in this chapter?

- From Fourier series to Fourier transform
- Existence of Fourier transform
- Fourier and Laplace transforms
- Time frequency relations and Fourier transform
- Spectral representation of periodic and aperiodic signals
- Modulation and signal transmission
- Convolution and Filtering

From the Fourier Series to the Fourier Transform

Aperiodic signal $x(t)$ is a periodic signal $\tilde{x}(t)$ with infinite period. From Fourier series $\tilde{x}(t)$ and limiting process we obtain:

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

where $x(t)$ is transformed into $X(\Omega)$ in the frequency-domain by

$$\textbf{Fourier transform:} \qquad X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

while $X(\Omega)$ is transformed into $x(t)$ in the time-domain by

$$\textbf{Inverse Fourier Transform:} \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

Aperiodic: $x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$ periodic of period T_0

Fourier Series:
$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t} \qquad \Omega_0 = \frac{2\pi}{T_0}$$

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-jn\Omega_0 t}dt$$

Define $X(\Omega_n) = T_0 X_n$ where $\{\Omega_n = n\Omega_0\}$, harmonic frequencies and $\Delta\Omega = 2\pi/T_0 = \Omega_0$ be frequency interval between harmonics then

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \frac{X(\Omega_n)}{T_0} e^{j\Omega_n t} = \sum_n X(\Omega_n)e^{j\Omega_n t}\frac{\Delta\Omega}{2\pi}$$

$$X(\Omega_n) = \int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-j\Omega_n t}dt$$

As $T_0 \to \infty$ then $\Delta\Omega \to d\Omega$, the line spectrum becomes denser, i.e. the lines in the line spectrum get closer, the sum becomes an integral, and $\Omega_n = n\Omega_0 = n\Delta\Omega \to \Omega$ so that in the limit we obtain

$$\text{Inverse FT: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

$$\text{FT: } X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

Existence of the Fourier Transform *The Fourier transform*

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

of a signal $x(t)$ exists (i.e., we can calculate its Fourier transform via this integral) provided

- *$x(t)$ is absolutely integrable or the area under $|x(t)|$ is finite,*
- *$x(t)$ has only a finite number of discontinuities as well as maxima and minima.*

"It appears that almost nothing has a Fourier transform —nothing except practical communication signals. No signal amplitude goes to infinity and no signal lasts forever; therefore, no practical signal can have infinite area under it, and hence all have Fourier Transforms."(Prof. E.Craig)

Signals of practical interest have Fourier transforms and their spectra can be displayed using a **spectrum analyzer**

Fourier Transforms from the Laplace Transform

If ROC of $X(s) = \mathcal{L}[x(t)]$ contains the $j\Omega$-axis, then

$$\begin{aligned} \mathcal{F}[x(t)] &= \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ &= X(s)\,|_{s=j\Omega} \end{aligned}$$

Rules of Thumb for Computing the Fourier Transform of a Signal $x(t)$:

- *If $x(t)$ has a finite time support and in that support $x(t)$ is finite, $\mathcal{F}[x(t)]$ exists*

- *If $x(t)$ has a Laplace transform $X(s)$ with ROC containing $j\Omega$-axis, $X(\Omega) = X(s)|_{s=j\Omega}$.*

- *If $x(t)$ is periodic its Fourier transform is obtained from its Fourier series using delta functions*

- *If $x(t)$ is none of the above, if it has discontinuities (e.g., $x(t) = u(t)$), or it is has discontinuities and it is not finite energy (e.g., $x(t) = \cos(\Omega_0 t)u(t)$) or it has possible discontinuities in the frequency domain even though it has finite energy (e.g., $x(t) = sinc(t)$) use properties of the Fourier transform.*

Example Fourier transform using Laplace for

$$\begin{aligned} &(a) \quad x_1(t) = u(t) \\ &(b) \quad x_2(t) = e^{-2t}u(t) \\ &(c) \quad x_3(t) = e^{-|t|} \end{aligned}$$

(a) $\mathcal{L}[x_1(t)] = X_1(s) = 1/s$, ROC $= \{s = \sigma + j\Omega : \sigma > 0, -\infty < \Omega < \infty\}$, Laplace transform cannot be used to find $\mathcal{F}[x_1(t)]$
(b) $\mathcal{L}[x_2(t)] = X_2(s) = 1/(s+2)$, ROC $= \{s = \sigma + j\Omega : \sigma > -2, -\infty < \Omega < \infty\}$ containing $j\Omega$-axis, then

$$X_2(\Omega) = \frac{1}{s+2}\,|_{s=j\Omega} = \frac{1}{j\Omega + 2}.$$

$$\text{(c)} \quad \mathcal{L}[x_3(t)] = X_3(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{2}{1-s^2}$$

ROC $= \{s = \sigma + j\Omega : -1 < \sigma < 1, -\infty < \Omega < \infty\}$ containing $j\Omega$-axis, then

$$X_3(\Omega) = X_3(s)|_{s=j\Omega} = \frac{2}{1-(j\Omega)^2} = \frac{2}{1+\Omega^2}$$

Inverse Proportionality of Time and Frequency

The support of $X(\Omega)$ is inversely proportional to the support of $x(t)$.
If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$ is

- *is contracted ($\alpha > 1$),*
- *is contracted and reflected ($\alpha < -1$),*
- *is expanded ($0 < \alpha < 1$),*
- *is expanded and reflected ($-1 < \alpha < 0$) or*
- *is reflected ($\alpha = -1$)*

$$x(\alpha t) \quad \Leftrightarrow \quad \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)$$

Frequency is inversely proportional to time:

- $x_1(t) = \delta(t)$, its support is only at $t = 0$, $X_1(\Omega) = 1, -\infty < \Omega < \infty$ with infinite support
- Opposite case: $x_2(t) = A$, $-\infty < t < \infty$, $X_2(\Omega) = 2\pi A\delta(\Omega)$ since the inverse Fourier transform is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\Omega)e^{j\Omega t}d\Omega = A$$

- Transition $x_2(t)$ to $x_1(t)$: consider $x_3(t) = A[u(t+\tau/2) - u(t-\tau/2)]$ with

$$X_3(s) = \frac{A}{s}\left[e^{s\tau/2} - e^{-s\tau/2}\right]$$

so that

$$X_3(\Omega) = X(s)|_{s=j\Omega} = A\tau\frac{\sin(\Omega\tau/2)}{\Omega\tau/2}$$

$A = 1/\tau$ as $\tau \to 0$ the pulse $x_3(t)$ becomes $\delta(t)$ and $X_3(\Omega)$ becomes unity
$\tau \to \infty$, $x_3(t) \quad \to \quad A\ X_3(\Omega) \quad \to \quad \delta(\Omega)$

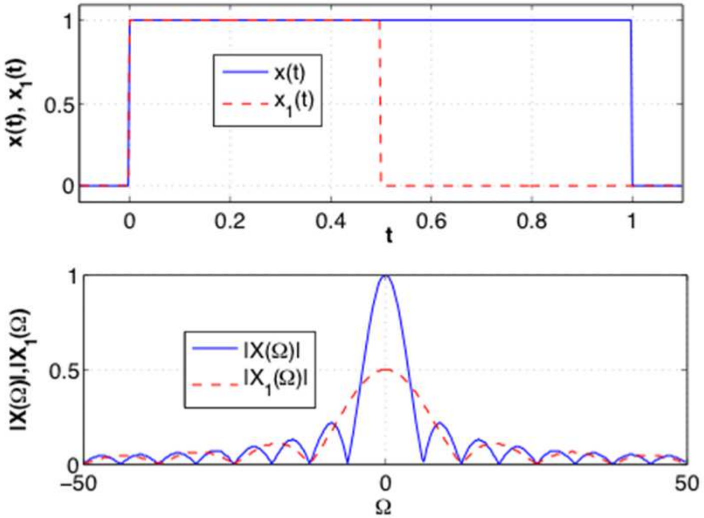
NOTE: $x(t) \quad \Leftrightarrow \quad X(\Omega)$ means to $x(t)$ in time domain corresponds a FT $X(\Omega)$ in the frequency domain. This is NOT an equality, far from it!

Example If $x(t) = u(t) - u(t-1)$, find FT of $x_1(t) = x(2t)$

$$X(s) = \frac{1-e^{-s}}{s} \qquad \text{ROC: whole s-plane}$$

$$X(\Omega) = \frac{1-e^{-j\Omega}}{j\Omega} = \frac{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})}{2j\Omega/2} = \frac{\sin(\Omega/2)}{\Omega/2}e^{-j\Omega/2}.$$

$$\begin{aligned} x_1(t) &= x(2t) = u(2t) - u(2t-1) = u(t) - u(t-0.5) \\ X_1(\Omega) &= \frac{1-e^{-j\Omega/2}}{j\Omega} = \frac{e^{-j\Omega/4}(e^{j\Omega/4} - e^{-j\Omega/4})}{j\Omega} \\ &= \frac{1}{2}\frac{\sin(\Omega/4)}{\Omega/4}e^{-j\Omega/4} = \frac{1}{2}X(\Omega/2) \end{aligned}$$

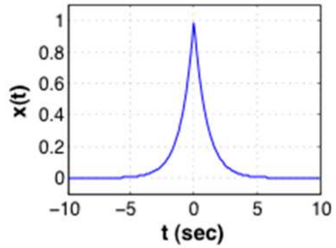


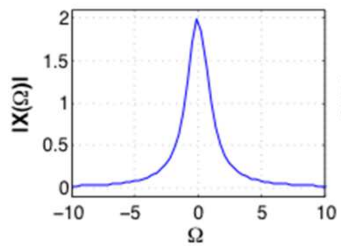
Example Apply the reflection property to find the Fourier transform of $x(t) = e^{-a|t|}$, $a > 0$

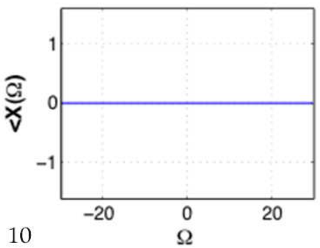
$$x(t) = e^{-at}u(t) + e^{at}u(-t) = x_1(t) + x_1(-t)$$

$$X_1(\Omega) = \frac{1}{s+a}\,|_{s=j\Omega} = \frac{1}{j\Omega + a}$$

$$\mathcal{F}[x_1(-t)] = \frac{1}{-j\Omega + a} \quad \Rightarrow X(\Omega) = \frac{1}{j\Omega + a} + \frac{1}{-j\Omega + a} = \frac{2a}{a^2 + \Omega^2}$$







Duality
To the Fourier transform pair

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

corresponds the following dual Fourier transform pair

$$X(t) \quad \Leftrightarrow \quad 2\pi x(-\Omega)$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\rho) e^{j\rho t} d\rho$$

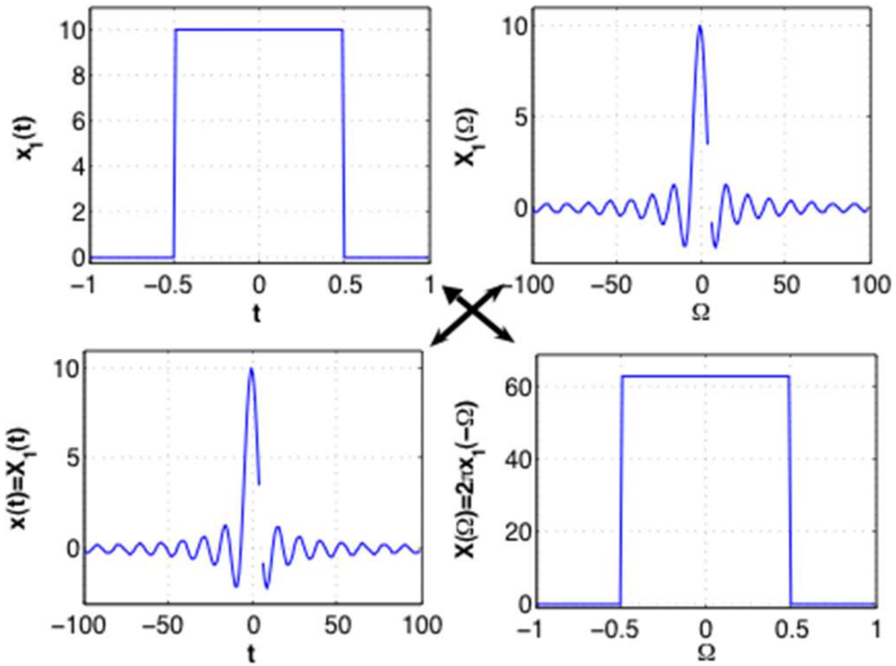
replace t by $-\Omega$ and multiply by 2π:

$$2\pi x(-\Omega) \quad = \quad \int_{-\infty}^{\infty} X(\rho) e^{-j\rho\Omega} d\rho = \int_{-\infty}^{\infty} X(t) e^{-j\Omega t} dt = \mathcal{F}[X(t)]$$

ρ and t are dummy variables inside the integral

- Example: Fourier transform pair

$$\begin{aligned} A\delta(t) \quad &\Leftrightarrow \quad A \\ A \quad &\Leftrightarrow \quad 2\pi A\delta(-\Omega) = 2\pi A\delta(\Omega) \end{aligned}$$



Application of duality to find Fourier transform of $x(t) = 10sinc(0.5t)$. Notice that $X(\Omega) = 2\pi x_1(\Omega) \approx 6.28x_1(\Omega) = 63.8[u(\Omega + 0.5) - u(\Omega - 0.5)]$

Example Find the Fourier transform of $x(t) = A\cos(\Omega_0 t)$ using duality
Consider the following Fourier pair

$$\delta(t - \rho_0) + \delta(t + \rho_0) \quad \Leftrightarrow \quad e^{-j\rho_0\Omega} + e^{j\rho_0\Omega} = 2\cos(\rho_0\Omega)$$

$$2\cos(\rho_0 t) \quad \Leftrightarrow \quad 2\pi[\delta(-\Omega - \rho_0) + \delta(-\Omega + \rho_0)] = 2\pi[\delta(\Omega + \rho_0) + \delta(\Omega - \rho_0)]$$

Replacing ρ_0 by Ω_0 and canceling the 2 in both sides we have

$$x(t) = \cos(\Omega_0 t) \qquad \Leftrightarrow \qquad X(\Omega) = \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)]$$

indicating that it only exists at $\pm\Omega_0$

Spectral Representation — Unification of the spectral representation of both periodic and aperiodic signals

Signal Modulation

Frequency shift: *If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair*

$$x(t)e^{j\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega - \Omega_0)$$

Modulation: *The Fourier transform of the* **modulated signal**

$$x(t)\cos(\Omega_0 t)$$

is given by

$$0.5\,[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

i.e., $X(\Omega)$ is shifted to frequencies Ω_0 and $-\Omega_0$, and multiplied by 0.5.

Remarks

- Amplitude modulation: consists in multiplying message $x(t)$ by a sinusoid of frequency higher than the maximum frequency of the incoming signal

$$\text{Modulated signal} \quad x(t)\cos(\Omega_0 t) = 0.5[x(t)e^{j\Omega_0 t} + x(t)e^{-j\Omega_0 t}]$$
$$\mathcal{F}[x(t)\cos(\Omega_0 t)] = 0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_o)]$$

  Modulation shifts the frequencies of $x(t)$ to frequencies around $\pm\Omega_0$

- Modulation using a sine, instead of a cosine, changes the phase of the Fourier transform of the incoming signal besides performing the frequency shift

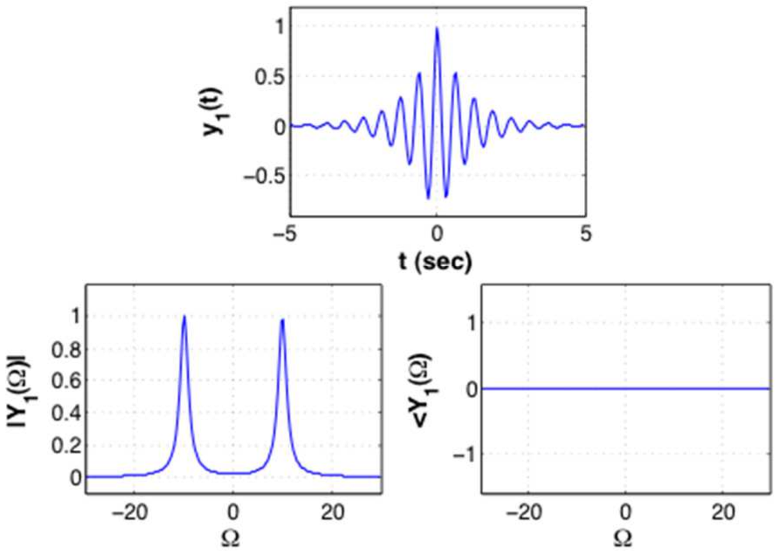
- According to eigenfunction property of LTI systems, modulation systems are not LTI.

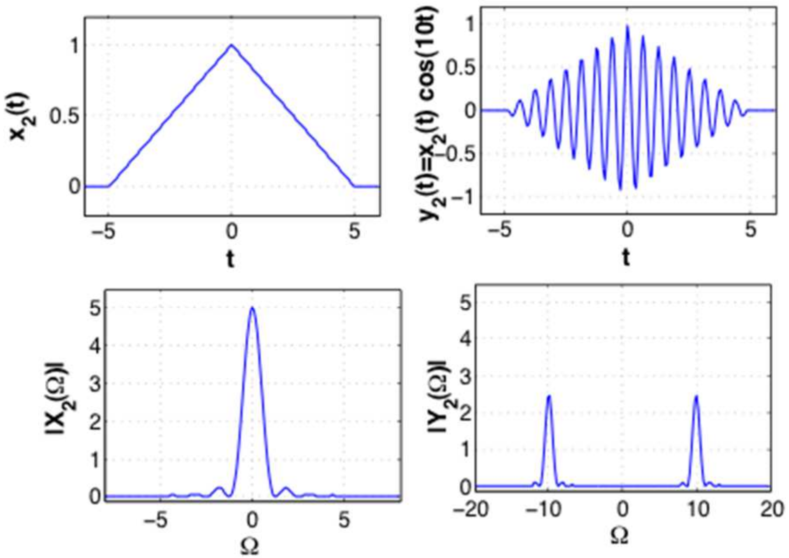
Example Modulate a carrier $\cos(10t)$ with:

1. $x_1(t) = e^{-|t|}$, $-\infty < t < \infty$. $x_1(t)$ is low-pass signal see spectrum before
2. $x_2(t) = 0.2[r(t+5) - 2r(t) + r(t+5)]$.

The modulated signals are

$$(i) \quad y_1(t) = x_1(t)\cos(10t) = e^{-|t|}\cos(10t), \qquad -\infty < t < \infty$$
$$(ii) \quad y_2(t) = x_2(t)\cos(10t) = 0.2[r(t+5) - 2r(t) + r(t+5)]\cos(10t)$$





Why Modulation? Modulation changes frequency content of a message from its baseband frequencies to higher frequencies making its transmission over the airwaves possible

Music ($0 \leq f \leq 22$KHz), and speech ($100 \leq f \leq 5$KHz) relatively low frequency signals requiring an antenna of length

$$\frac{\lambda}{4} = \frac{3 \times 10^8}{4f} \quad \text{meters}$$

$$\text{if } f = 30\text{KHz} \Rightarrow \text{ length of antenna } 2.5\text{km} \approx 1.5\text{miles}$$

thus need to increase baseband frequencies.

Fourier Transform of Periodic Signals

A periodic signal $x(t)$ of period T_0:

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \qquad \Leftrightarrow \qquad X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

Fourier series of $x(t)$:

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \qquad \Omega_0 = 2\pi/T_0$$

$$X(\Omega) = \sum_k \mathcal{F}[X_k e^{jk\Omega_0 t}] = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

Remarks

- $|X(\Omega)|$ vs Ω, the Fourier magnitude spectrum of periodic $x(t)$ is analogous to its line spectrum
- Direct computation

$$\mathcal{F}[\cos(\Omega_0 t)] = \mathcal{F}[0.5e^{j\Omega_0 t} + 0.5e^{-j\Omega_0 t}] = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

$$\begin{aligned}\mathcal{F}[\sin(\Omega_0 t)] &= \mathcal{F}\left[\frac{0.5}{j}e^{j\Omega_0 t} - \frac{0.5}{j}e^{-j\Omega_0 t}\right] = \frac{\pi}{j}\delta(\Omega - \Omega_0) - \frac{\pi}{j}\delta(\Omega + \Omega_0) \\ &= \pi e^{-j\pi/2}\delta(\Omega - \Omega_0) + \pi e^{j\pi/2}\delta(\Omega + \Omega_0)\end{aligned}$$

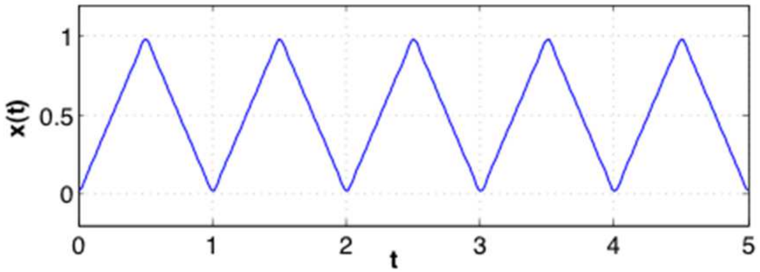
Example Triangular pulses $x(t)$ with a period

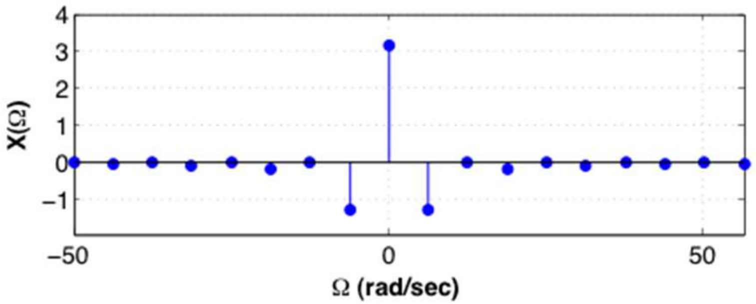
$$x_1(t) = r(t) - 2r(t-0.5) + r(t-1),\ \Omega_0 = 2\pi$$

find $X(\Omega)$

$$\begin{aligned}
X_1(s) &= \frac{1}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right) = \frac{e^{-0.5s}}{s^2}\left(e^{0.5s} - 2 + e^{-0.5s}\right) \\
\text{FS coefficients} \quad X_k &= \frac{1}{T_0} X_1(s)|_{s=j2\pi k} = \frac{1}{(j2\pi k)^2} 2(\cos(\pi k) - 1)e^{-j\pi k} \\
&= (-1)^{(k+1)} \frac{\cos(\pi k) - 1}{2\pi^2 k^2} = (-1)^k \frac{\sin^2(\pi k/2)}{\pi^2 k^2} \\
X(0) &= 0.5
\end{aligned}$$

$$\text{FT}: \quad X(\Omega) = 2\pi X_0 \delta(\Omega) + \sum_{k=-\infty, \neq 0}^{\infty} 2\pi X_k \delta(\Omega - 2k\pi)$$





Parseval's Energy Conservation *A finite-energy signal $x(t)$, with Fourier transform $X(\Omega)$, its energy is*

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

$|X(\Omega)|^2$ *energy density : energy per frequency*
$|X(\Omega)|^2$ *vs* Ω*: energy spectrum of $x(t)$, energy of the signal distributed over frequency*

Example Is $\delta(t)$ a finite energy signal?
Using Parseval's result: $\mathcal{F}\delta(t) = 1$ for all frequencies then its energy is infinite
In time-domain:

$$p_\Delta(t) = \frac{1}{\Delta}[u(t + \Delta/2) - u(t - \Delta/2)] \quad \Rightarrow \quad \delta(t) \text{as } \Delta \to 0, \text{ unit area}$$
$$p^2_\Delta(t) = \frac{1}{\Delta^2}[u(t + \Delta/2) - u(t - \Delta/2)] \quad \Rightarrow \quad \delta^2(t) \text{as } \Delta \to 0, \text{ infinite area } 1/\Delta$$

$\delta(t)$ is not finite energy

Symmetry of Spectral Representations
If $X(\Omega)$ is FT of real-valued signal $x(t)$, periodic or aperiodic then

- *Magnitude $|X(\Omega)|$ even function of Ω:*

$$|X(\Omega)| = |X(-\Omega)|$$

- *Phase $\angle X(\Omega)$ odd function of Ω:*

$$\angle X(\Omega) = -\angle X(-\Omega)$$

$|X(\Omega)|$ *vs* Ω **Magnitude Spectrum**
$\angle X(\Omega)$ *vs* Ω **Phase Spectrum**
$|X(\Omega)|^2$ *vs* Ω **Energy/Power Spectrum.**

Remarks

- If signal is complex, the above symmetry will not hold
- Meaning of "negative" frequencies:
  - only positive frequencies exist and can be measured,
  - spectrum of a real signal requires negative frequencies

Example MATLAB to compute Fourier transform of

$$
\begin{array}{ll}
(a) & x(t) = u(t) - u(t-1) \\
(b) & x(t) = e^{-t}u(t)
\end{array}
$$

For $x(t) = e^{-t}u(t)$

```
%%%%%%%%%%%%%%%%%%%%%%
% Example 5.11
%%%%%%%%%%%%%%%%%%%%%%
sym t
x2=exp(-t)*heaviside(t);
X2=fourier(x2)
X2m=sqrt((real(X2))^2+(imag(X2))^2; % magnitude
X2p=imag(log(X2); % phase
```

Magnitude:

$$|X_2(\Omega)| = \sqrt{\mathcal{R}e[X_2(\Omega)]^2 + \mathcal{I}m[X_2(\Omega)]^2}.$$

$$\log(X_2(\Omega)) = \log(|X_2(\Omega)|e^{j\angle X_2(\Omega)} = \log(|X_2(\Omega)| + j\angle X_2(\Omega)$$

so that

$$\angle X_2(\Omega) = \mathcal{I}m[\log(X_2(\Omega))].$$

$$z(t) = x_1(t+0.5) = u(t+0.5) - u(t-0.5)$$

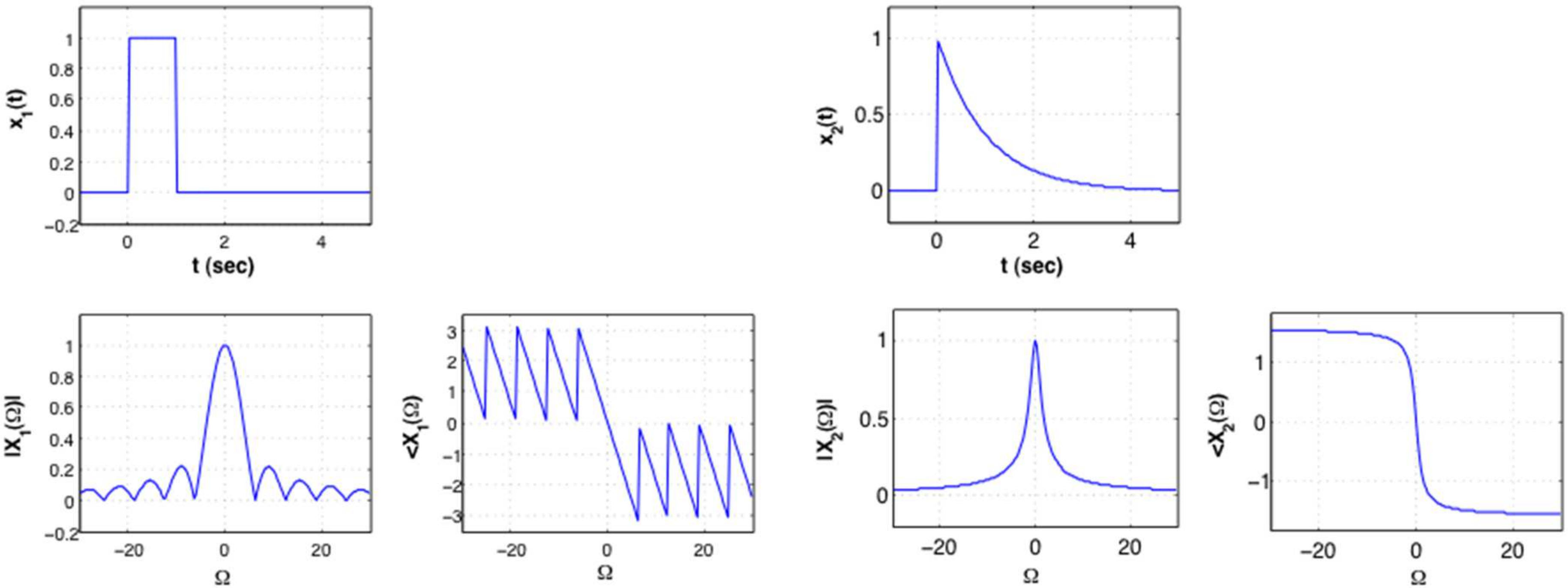
$$Z(\Omega) = \frac{\sin(\Omega/2)}{\Omega/2} \qquad \text{real-valued}$$

$$\angle Z(\Omega) = \begin{cases} 0 & Z(\Omega) \geq 0 \\ \pm\pi & Z(\Omega) < 0 \end{cases}$$

$$z(t) = x_1(t+0.5) \;\Rightarrow\; Z(\Omega) = X_1(\Omega)e^{j0.5\Omega}$$

$$X_1(\Omega) = e^{-j0.5\Omega}Z(\Omega)$$

$$\angle X_1(\Omega) = \angle Z(\Omega) - 0.5\Omega = \begin{cases} -0.5\Omega & Z(\Omega) \geq 0 \\ \pm\pi - 0.5\Omega & Z(\Omega) < 0 \end{cases}$$



Convolution and Filtering *If $x(t)$ (periodic or aperiodic) is input to a stable LTI system with a frequency response $H(j\Omega) = \mathcal{F}[h(t)]$, $h(t)$ impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform*

$$Y(\Omega) = \mathcal{F}[(x * h)] = X(\Omega)\ H(j\Omega)$$

If $x(t)$ is periodic the output is also periodic with Fourier transform

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\ X_k\ H(jk\ \Omega_0)\delta(\Omega - k\Omega_0)$$

where X_k are the Fourier series coefficients of $x(t)$ and Ω_0 its fundamental frequency.

Eigenfunction property of LTI systems:
Aperiodic signals

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \Rightarrow \\ y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [X(\Omega)H(j\Omega)]\, e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\Omega)e^{j\Omega t}d\Omega \end{aligned}$$

Periodic signal of perio T_0

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k\delta(\Omega - k\Omega_0) \quad \Rightarrow \quad Y(\Omega) \quad = \quad X(\Omega)H(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0)\delta(\Omega - k\Omega_0)$$

output $y(t)$ is periodic

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{Y_k}_{X_k H(jk\Omega_0)} e^{jk\Omega_0 t}$$

Basics of Filtering

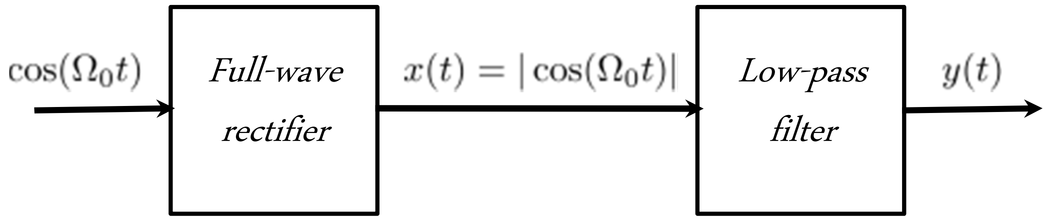
- Filtering consists in getting rid of undesirable components of a signal, e.g., noise $\eta(t)$ is added to a desired signal $x(t)$

$$y(t) = x(t) + \eta(t)$$

  Filter design: find $H(s) = B(s)/A(s)$ satisfying certain specifications to get rid of noise.

- Frequency discriminating filters keep the frequency components of a signal in a certain frequency band and attenuate the rest.

Example Obtain dc source of unity amplitude using a full-wave rectifier and a low-pass filter (it keeps only the low-frequency components)



FS coefficients:

$$X_0 = \frac{2}{\pi}$$
$$X_k = \frac{2(-1)^k}{\pi(1-4k^2)} \qquad k \neq 0$$

Filter out all harmonics and leave average component: ideal low-pass filter

$$H(j\Omega) = \begin{cases} A & -\Omega_0 < \Omega_c < \Omega_0, \quad \text{where} \quad \Omega_0 = 2\pi/T_0 = 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Ideal Filters

- Low-pass filter (keeps low-frequency components)

$$|H_{lp}(j\Omega)| = \begin{cases} 1 & -\Omega_1 \leq \Omega \leq \Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\angle H_{lp}(j\Omega) = -\alpha\Omega$$

- Band-pass filter (keeps middle frequency components)

$$|H_{bp}(j\Omega)| = \begin{cases} 1 & \Omega_1 \leq \Omega \leq \Omega_2 \quad \text{and} \quad -\Omega_2 \leq \Omega \leq -\Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

  linear phase in the passband

- High-pass filter (keeps high-frequency components)

$$|H_{hp}(j\Omega)| = \begin{cases} 1 & \Omega \geq \Omega_2 \quad \text{and} \quad \Omega \leq -\Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

  linear phase in the passband

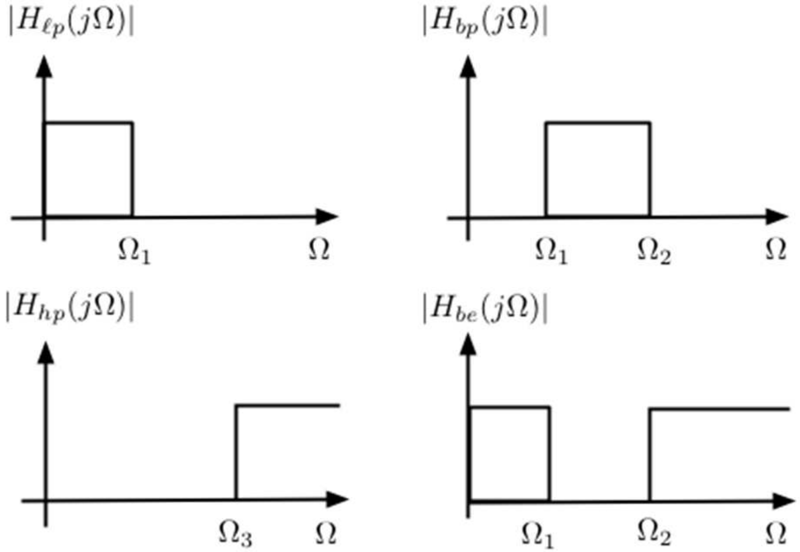
- Band-stop filter (attenuates middle frequency components)

$$|H_{bs}(j\Omega)| = 1 - |H_{bp}(j\Omega)|$$

- All-pass filter (keeps all frequency components, changes phase)

$$|H_{ap}(j\Omega)| = |H_{lp}(j\Omega)| + |H_{bp}(j\Omega)| + |H_{hp}(j\Omega)| = 1$$

- Multi-band filter: combination of the low-, band-, and high-pass filters



Example Gibb's phenomenon of Fourier series: ringing around discontinuities of periodic signals. Consider a periodic train of square pulses $x(t)$ of period T_0 displaying discontinuities at $kT_0/2$, for $k = \pm 1, \pm 2, \cdots$. Show Gibb's phenomenon is due to ideal low-pass filtering.
Ideal low-pass filter

$$H(j\Omega) = \begin{cases} 1 & -\Omega_c \leq \Omega \leq \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

Periodic signal $x(t)$, of fundamental frequency $\Omega_0 = 2\pi/T_0$, is

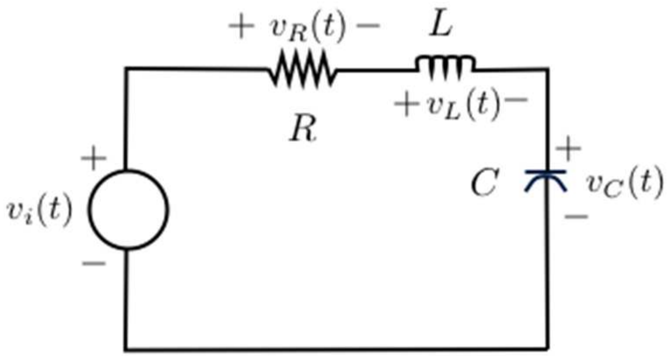
$$X(\Omega) = \mathcal{F}[x(t)] = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0)$$

Output of the filter:

$$\begin{aligned} x_N(t) &= \mathcal{F}^{-1}[X(\Omega)H(j\Omega)] = \mathcal{F}^{-1}\left[\sum_{k=-N}^{N} 2\pi X_k \delta(\Omega - k\Omega_0)\right] \\ &= [x * h](t) \qquad \text{cutoff frequency:} \quad N\Omega_0 < \Omega_c < (N+1)\Omega_0 \end{aligned}$$

convolution around the discontinuities of $x(t)$ causes ringing before and after them, independent of N

Example Obtain different filters from an RLC circuit by choosing different outputs. Let R= 1 Ω, L = 1 H, and C= 1 F, and IC=0



Low-pass Filter: Output $V_c(s)$

$$H_{lp}(s) = \frac{V_C(s)}{V_i(s)} = \frac{1}{s^2 + s + 1}$$

- input a dc source (frequency $\Omega = 0$), inductor short circuit, capacitor open circuit, so $V_c(s) = V_i(s)$
- input of very high frequency, $\Omega \to \infty$, inductor open circuit, capacitor short circuit $V_c(s) = V_i(s) = 0$

High-pass Filter: Output $V_L(s)$

$$H_{hp}(s) = \frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2 + s + 1}$$

- dc input (frequency zero), inductor is short circuit $V_L(s) = 0$
- input of very high frequency, $\Omega \to \infty$, inductor is open circuit $V_L(s) = V_i(s)$

Band-pass Filter: Output $V_R(s)$

$$H_{bp}(s) = \frac{V_R(s)}{V_i(s)} = \frac{s}{s^2 + s + 1}$$

- For zero frequency, capacitor is open circuit so voltage across the resistor is zero
- For very high frequency, inductor is open circuit, voltage across resistor is zero
- For some middle frequency, serial LC combination resonates (zero impedance) maximum voltage across resistor

Band-stop Filter: Output voltage across inductor and the capacitor

$$H_{bs}(s) = \frac{s^2 + 1}{s^2 + s + 1}$$

- At low and high frequencies, LC is open-circuit, $V_{LC}(s) = V_i(s)$
- At the resonance frequency $\Omega_r = 1$ the impedance of the LC connection is zero, so the output voltage is zero

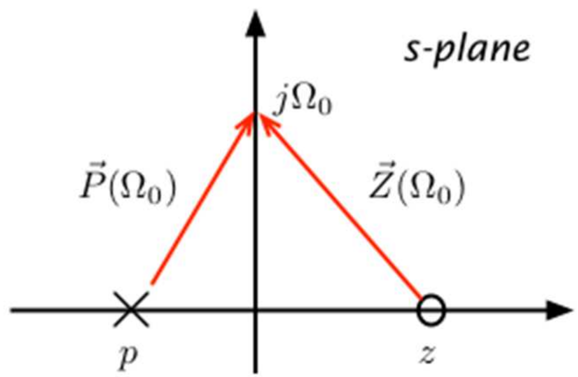
Frequency Response from Poles and Zeros

For a filter with transfer function

$$H(s) = \frac{\prod_i (s - z_i)}{\prod_k (s - p_k)} \quad \textit{where} \quad z_i, p_k \quad \textit{are zeros and poles of} \quad H(s)$$

with vectors $\vec{Z}_i(\Omega)$ *and* $\vec{P}_k(\Omega)$, *going from each of the zeros and poles to the frequency at which we are computing the magnitude and phase response in the* $j\Omega$*-axis, gives*

$$\begin{aligned} H(j\Omega) = H(s)|_{s=j\Omega} &= \frac{\prod_i \vec{Z}_i(\Omega)}{\prod_k \vec{P}_k(\Omega)} \\ &= \underbrace{\frac{\prod_i |\vec{Z}_i(\Omega)|}{\prod_k |\vec{P}_k(\Omega)|}}_{|H(j\Omega)|} \underbrace{e^{j\left[\sum_i \angle(\vec{Z}_i(\Omega)) - \sum_k \angle(\vec{P}_k(\Omega))\right]}}_{e^{j\angle H(j\Omega)}} \end{aligned}$$



Example RC circuit in series with voltage source $v_i(t)$. Choose the output to obtain low-pass and high-pass filters and use the poles and zeros of the transfer functions to determine their frequency responses. Let $R = 1\ \Omega$, $C = 1$ F and the initial conditions be zero.

Low-pass filter:

$$\begin{aligned} H(s) &= \frac{V_C(s)}{V_i(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1+s} \\ H(j\Omega) &= \frac{1}{1+j\Omega} = \frac{1}{\vec{P}(\Omega)} \end{aligned}$$

Geometrically

$$\begin{aligned} &\Omega = 0 \qquad \vec{P}(0) = 1e^{j0} H(j0) = 1e^{j0} \\ &\Omega = 1 \qquad \vec{P}(1) = \sqrt{2}e^{j\pi/4} \quad \Rightarrow \quad H(j1) = 0.707e^{-j\pi/4} \\ &\Omega = \infty \qquad \vec{P}(\infty) = \infty\ e^{j\pi/2} \quad \Rightarrow \quad H(j\infty) = 0e^{-j\pi/2} \end{aligned}$$

High-pass filter:

$$\begin{aligned} H(s) &= \frac{V_r(s)}{V_s(s)} = \frac{CRs}{CRs+1} = \frac{s}{s+1} \\ H(j\Omega) &= \frac{j\Omega}{1+j\Omega} = \frac{\vec{Z}(\Omega)}{\vec{P}(\Omega)} \end{aligned}$$

$$\begin{aligned} &\Omega = 0 \qquad \vec{P}(0) = 1e^{j0} \qquad \vec{Z}(0) = 0e^{j\pi/2} \qquad H(j0) = \frac{\vec{Z}(0)}{\vec{P}(0)} = 0e^{j\pi/2} \\ &\Omega = 1 \qquad \vec{P}(1) = \sqrt{2}e^{j\pi/4} \qquad \vec{Z}(1) = 1e^{j\pi/2} \qquad H(j1) = \frac{\vec{Z}(1)}{\vec{P}(1)} = 0.707e^{j\pi/4} \\ &\Omega = \infty \qquad \vec{P}(\infty) = \infty\ e^{j\pi/2} \qquad \vec{Z}(\infty) = \infty\ e^{j\pi/2} \qquad H(j\infty) = \frac{\vec{Z}(\infty)}{\vec{P}(\infty)} = 1e^{j0} \end{aligned}$$

Remarks

- Poles create "hills" at frequencies in the $j\Omega$-axis in front of their imaginary parts.
- Zeros create "valleys" at the frequencies in the $j\Omega$-axis in front of their imaginary parts

Example Use MATLAB to find and plot the poles and zeros and the corresponding magnitude and phase frequency responses of:
(a) A second-order band-pass and a high-pass filters realized using an RLC series connection (R=1, L=1, C=1)
(b) An all-pass filter with a transfer function

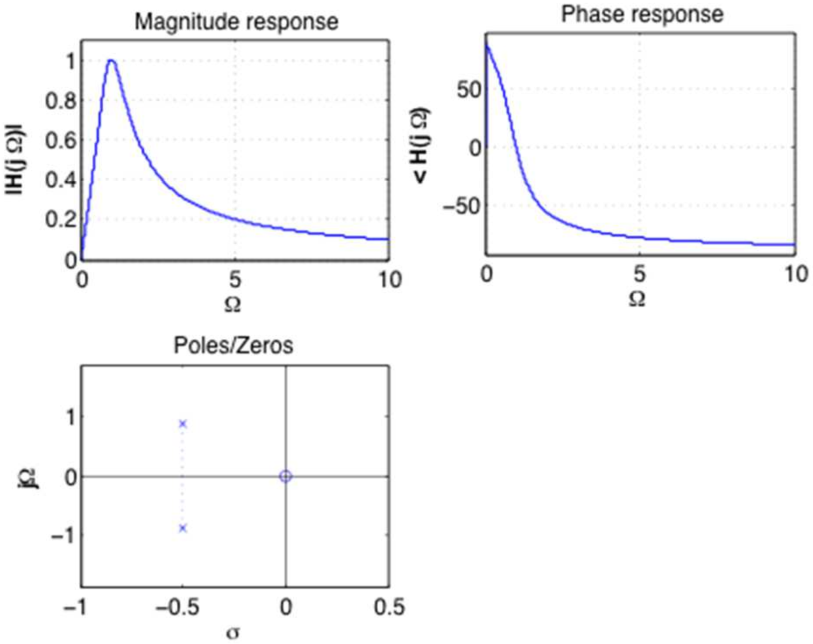
$$H(s) = \frac{s^2 - 2.5s + 1}{s^2 + 2.5s + 1}$$

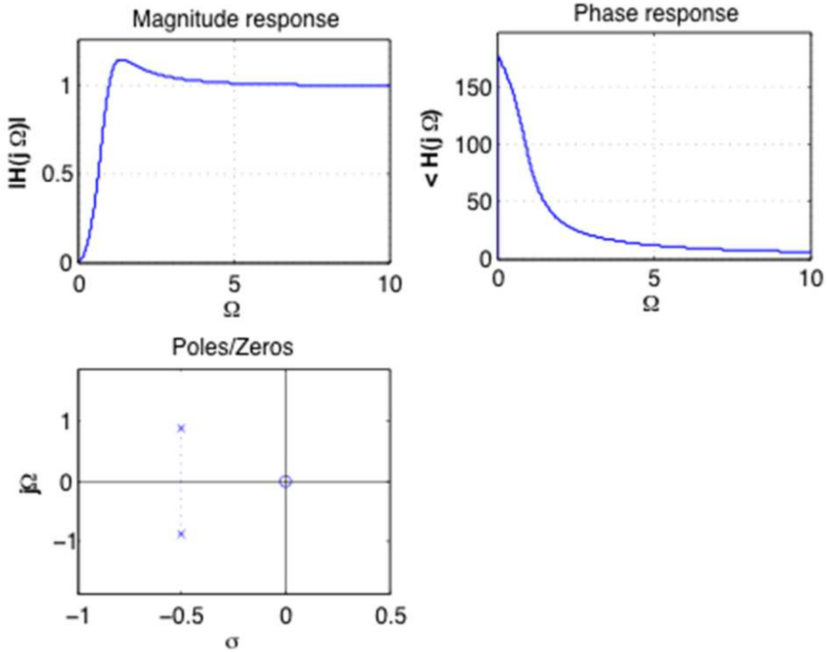
(a) From previous example,

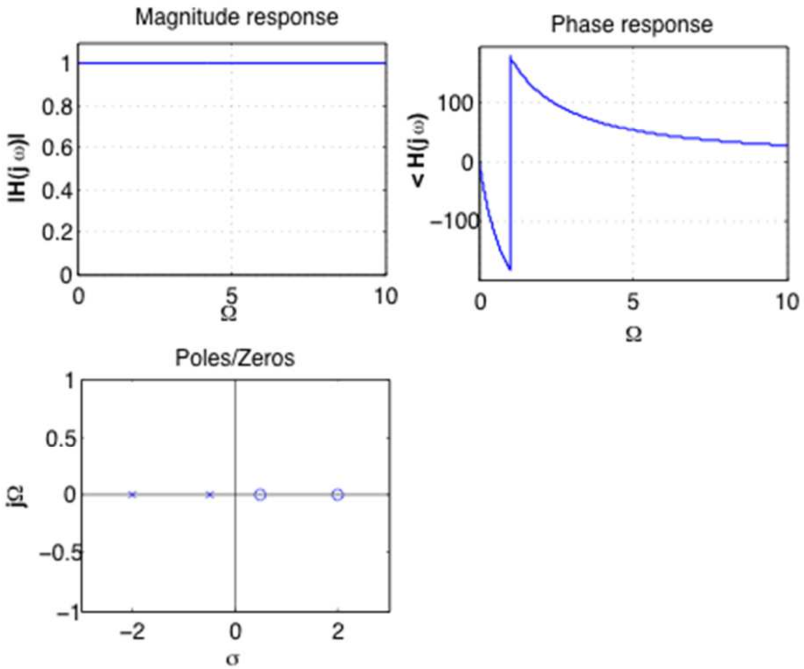
$$H_{bp}(s) = \frac{s}{s^2 + s + 1}$$
$$H_{hp}(s) = \frac{s^2}{s^2 + s + 1}$$

```
%%%%%%%%%%%%%%%%%%%%%
% Example 5.18 --- Frequency response
%%%%%%%%%%%%%%%%%%%%%
n=[0 1 0]; % numerator coefficients -- bandpass
% n=[1 0  0]; % numerator coefficients -- highpass
d=[1 1 1]; % denominator coefficients
wmax=10; % maximum frequency
[w,Hm,Ha]=freqresp_s(n,d,wmax); % frequency response
splane(n,d) % plotting of poles and zeros
```

```
function [w,Hm,Ha]=freqresp_s(b,a,wmax)
w=0:0.01:wmax;
H=freqs(b,a,w);
Hm=abs(H); % magnitude
Ha=angle(H)*180/pi; % phase in degrees
```


Magnitude response
|H(j Ω)|
Ω
Phase response
< H(j Ω)
Ω
Poles/Zeros
jΩ
σ


Magnitude response
|H(j Ω)|
Ω
Phase response
< H(j Ω)
Ω
Poles/Zeros
jΩ
σ


Magnitude response
|H(j ω)|
Ω
Phase response
< H(j ω)
Ω
Poles/Zeros
jΩ
σ

The Spectrum Analyzer— Device that measures the spectral characteristics of a signal

- Implemented as a bank of narrow-band band-pass filters with fixed bandwidths covering the desired frequencies (used for the audio range of the spectrum)
  Input $x(t)$, output of one of bandpass filter with very narrow bandwidth $\Delta\Omega$:

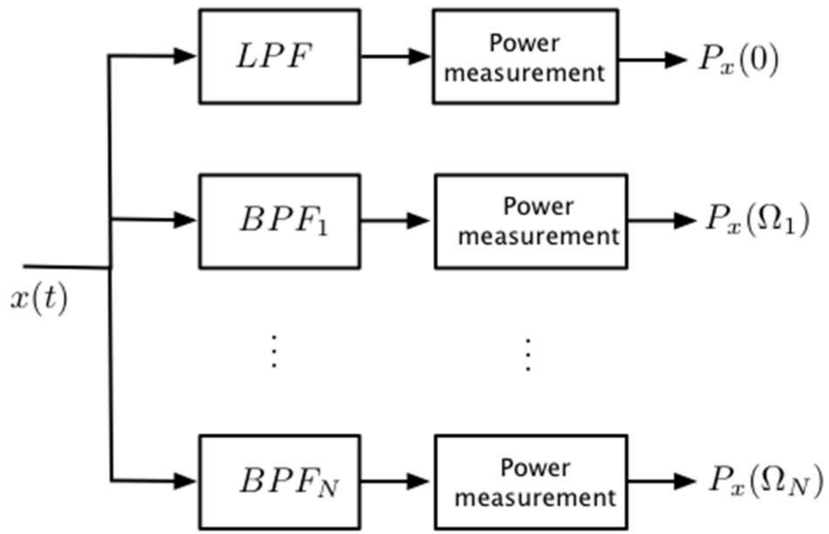
$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{\Omega_0-0.5\Delta\Omega}^{\Omega_0+0.5\Delta\Omega} X(\Omega)e^{j\Omega t}d\Omega \\ &\approx \frac{1}{2\pi}\Delta\Omega \;\; X(\Omega_0)e^{j\Omega_0 t} \end{aligned}$$

  mean square of this signal

$$\frac{1}{T}\int_T |y(t)|^2 dt = \left(\frac{\Delta\Omega}{2\pi}\right)^2 \; |X(\Omega_0)|^2$$

  proportional to the power or the energy of the signal in $\Omega_0 \pm \Delta\Omega$. A similar computation can be done at each of the frequencies of the input signal

- Radio frequency spectrum analyzers resemble an AM demodulator



What have we accomplished?

- Unification of frequency representation of periodic and aperiodic signals
- Frequency response of LTI systems
- Duality in time and frequency
- Convolution and Filtering
- Connection of Fourier series and Laplace transform
- Inverse time frequency relation

Where do we go from here?

- Application of Laplace analysis and transient response
- Application of Fourier analysis and steady state response
- Filter design
- Application of time-frequency relation in sampling theory