Signals and Systems Using MATLAB

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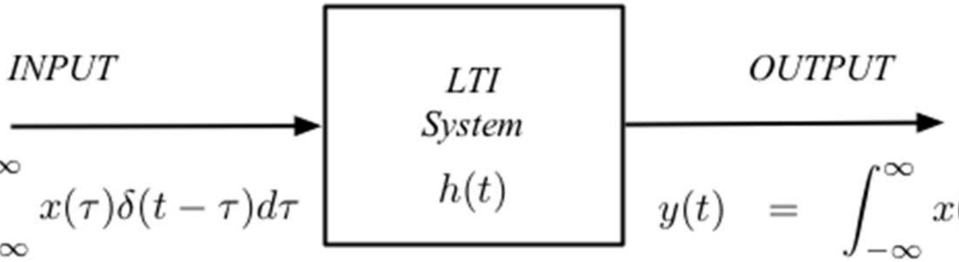
Chapter 3 - The Laplace Transform

What is in this chapter?

- Definition of Laplace transform
- Properties of Laplace transform
- Inverse Laplace transform
- Convolution sum and Laplace
- Analysis of LTI systems using Laplace transform
- Convolution integral
- System interconnection

Signals and LTI systems



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \end{aligned}$$

Impulse response:

$$\text{Input: } \delta(t), \ \text{IC} = 0 \quad \Rightarrow \quad h(t)$$

Linearity:

$$x(\tau)\delta(t) \quad \Rightarrow \quad x(\tau)h(t)$$

Time-invariance:

$$x(\tau)\delta(t-\tau) \quad \Rightarrow \quad x(\tau)h(t-\tau)$$

Linearity (superposition):

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}_{x(t)} \quad \Rightarrow \quad \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{y(t)}$$

Eigenfunctions of LTI Systems

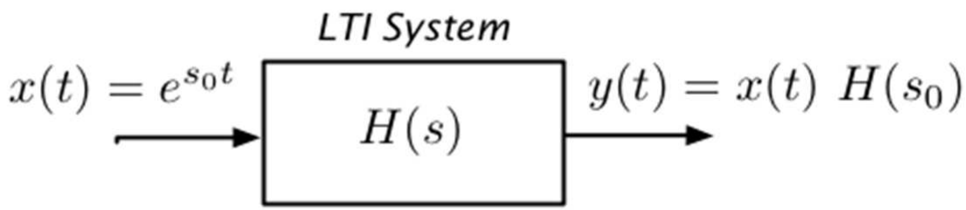
LTI system with $h(t)$ (impulse response), input

$$x(t) = e^{s_0 t} \qquad s_0 = \sigma_0 + j\Omega_0 \qquad -\infty < t < \infty$$

convolution integral:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau = \underbrace{e^{s_0 t}}_{x(t)} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-\tau s_0}d\tau}_{H(s_0)} \end{aligned}$$

Same exponential at the input appears at the output, $x(t) = e^{s_0 t}$ is called an **eigenfunction**



Laplace transform

The two-sided Laplace transform of a continuous–time function $f(t)$ is

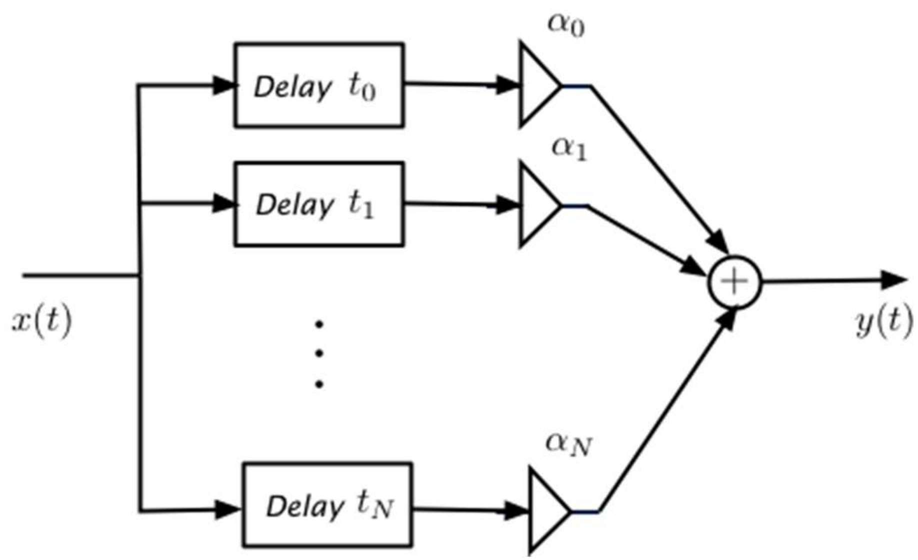
$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \qquad s \in ROC$$

where the variable $s = \sigma + j\Omega$, with Ω frequency in rad/sec and σ a damping factor. ROC stands for the region of convergence, i.e., where the integral exists.

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \qquad \sigma \in \text{ROC}$$

Example Wireless communications: “multi-path” effect



$$y(t) = \alpha_0 x(t - t_0) + \alpha_1 x(t - t_1) + \cdots + \alpha_N x(t - t_N)$$

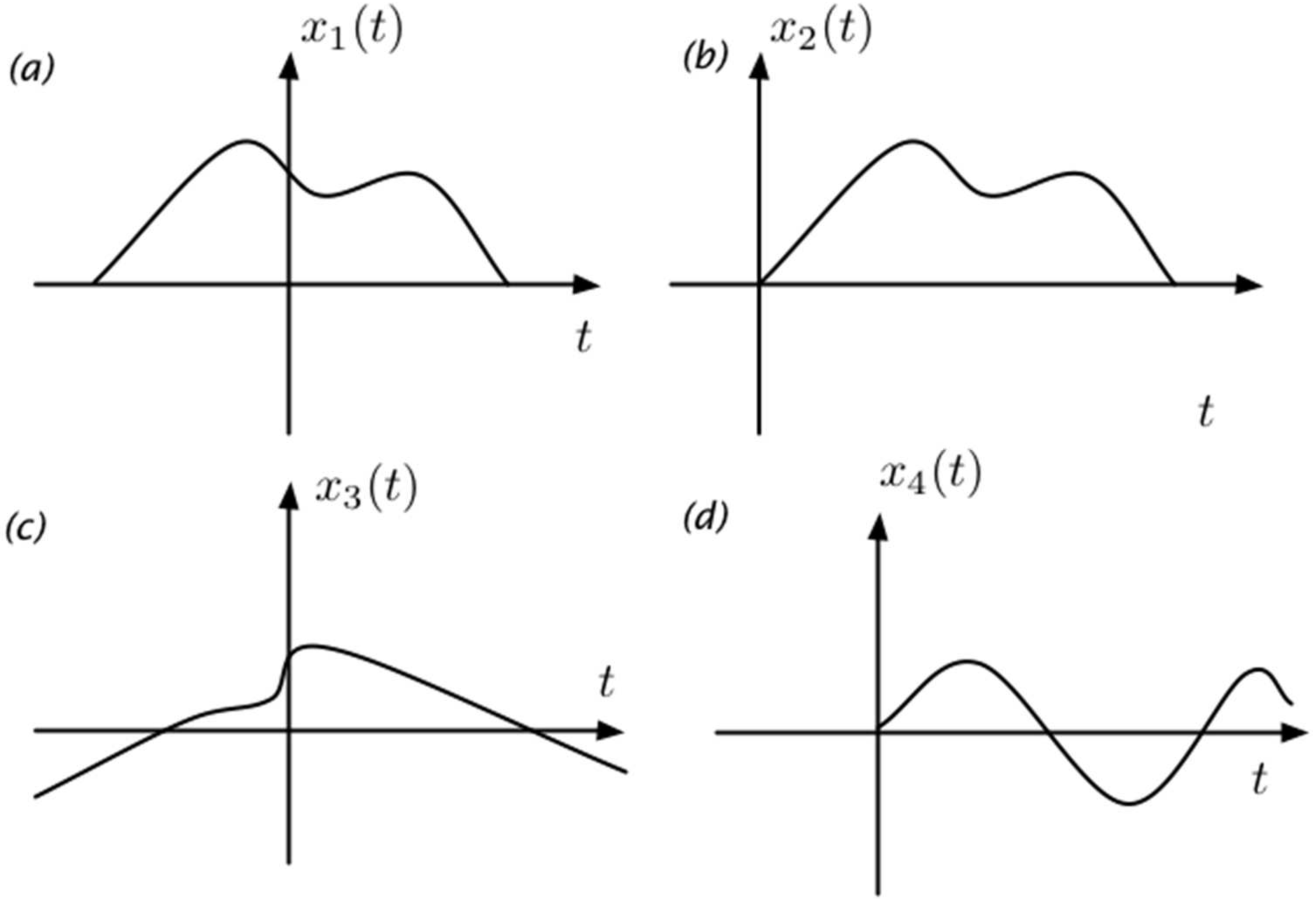
Response of multi-path system to $x(t) = e^{st}$ is $y(t) = x(t)H(s)$ so

$$x(t)H(s) = x(t)\left[\alpha_0 e^{-st_0} + \cdots + \alpha_N e^{-st_N}\right]$$

Channel system function:

$$H(s) = \alpha_0 e^{-st_0} + \cdots + \alpha_N e^{-st_N}$$

Notice: time shifts became exponentials in Laplace domain



Examples of different types of signals:
(a) non-causal finite support signal $x_1(t)$,
(b) causal finite support signal $x_2(t)$,
(c) non-causal infinite support signal $x_3(t)$, *and*
(d) causal infinite-support $x_4(t)$.

Poles and Zeros

Rational function

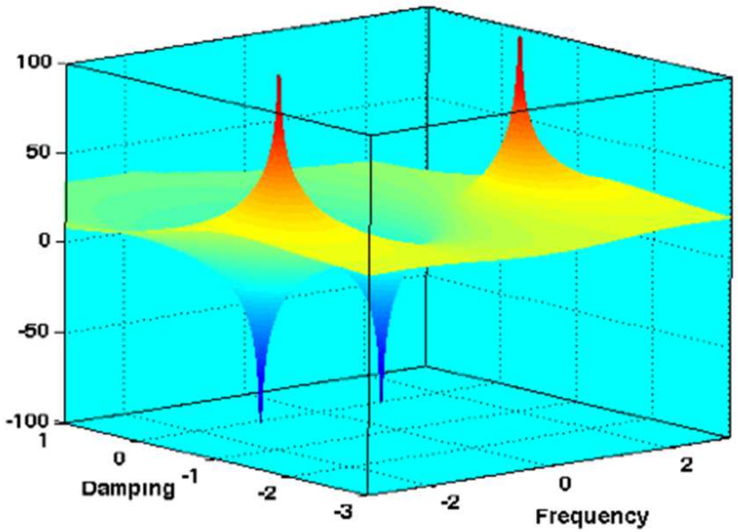
$$F(s) = \mathcal{L}[f(t)] = N(s)/D(s)$$

zeros of $F(s)$*: values of s that make* $F(s) = 0$
poles of $F(s)$ *values of s that make* $F(s) \to \infty$.

Example

$$F(s) = \frac{2(s^2+1)}{s^2+2s+5} = \frac{2(s+j)(s-j)}{(s+1)^2+4} = \frac{2(s+j)(s-j)}{(s+1+2j)(s+1-2j)}$$

zeros $s_{1,2} = \pm j$, roots of $N(s) = 0$, $F(\pm j) = 0$
poles $s_{1,2} = -1 \pm 2j$, roots of $D(s) = 0$, $F(-1 \pm 2j) \to \infty$



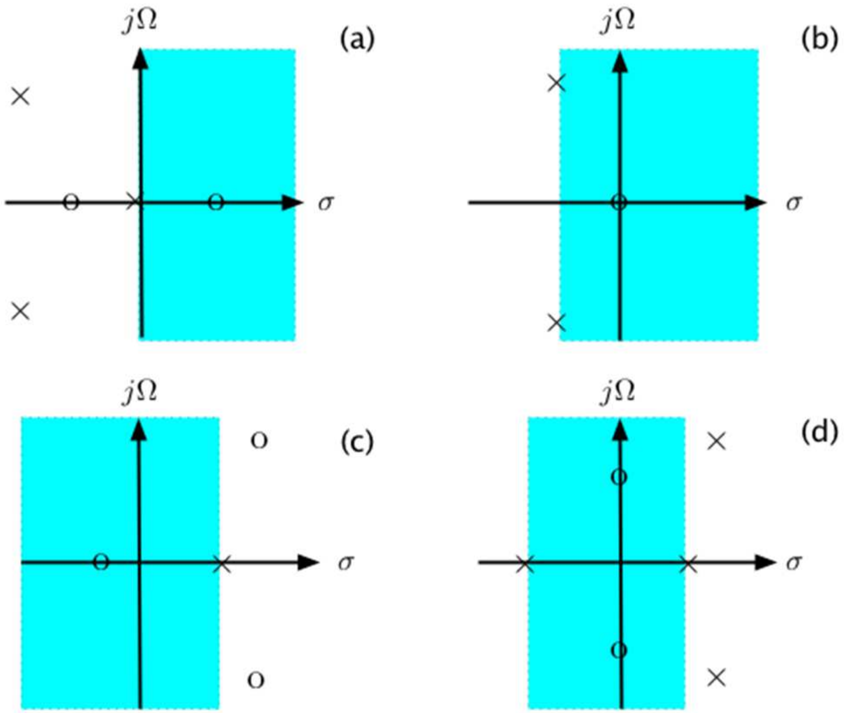
Three

Poles and Region of Convergence

ROC: values of σ such that

$$\left|\int_{-\infty}^{\infty} x(t)e^{-st}dt\right| \leq \int_{-\infty}^{\infty} |x(t)||e^{-(\sigma+j\Omega)t}|dt = \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$$

- No poles are included in the ROC
- The ROC is a plane parallel to the $j\Omega$-axis



ROC for (a) causal signal with poles with $\sigma_{max} = 0$; (b) causal signal with poles with $\sigma_{max} < 0$; (c) anti-causal signal with poles with $\sigma_{min} > 0$; (d) two-sided or noncausal signal where ROC is bounded by poles (poles on left-hand plane give causal component and poles on the right-hand s-plane give the anti-causal component of the signal)

- Not all rational functions have finite number of poles/ zeros

$$P(s) = \frac{1}{s}\left(e^s - e^{-s}\right).$$

Possible pole $s = 0$
Zeros: let $e^s - e^{-s} = 0$, or

$$e^{2s} = 1 = e^{j2\pi k} \qquad k = 0, \pm 1, \pm 2, \cdots$$

zeros: $s_k = j\pi k,\ k = 0, \pm 1, \pm 2, \cdots$.
For $k = 0$, zero cancels pole at zero. $P(z)$ has infinite number of zeros, no poles.

The Laplace transform of a

- *Finite support function, i.e., $f(t) = 0$ for $t < t_1$ and $t > t_2$, for $t_1 < t_2$,*

$$\mathcal{L}[f(t)] = \mathcal{L}\left[f(t)[u(t-t_1) - u(t-t_2)]\right] \qquad \textit{whole s-plane}$$

- *Causal function, i.e., $f(t) = 0$ for $t < 0$, is*

$$\mathcal{L}[f(t)u(t)] \qquad \mathcal{R}_c = \{(\sigma, \Omega) : \sigma > \max\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- *Anticausal function, i.e., $f(t) = 0$ for $t > 0$, is*

$$\mathcal{L}[f(t)u(-t)] \qquad \mathcal{R}_{ac} = \{(\sigma, \Omega) : \sigma < \min\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- *Non-causal, i.e., $f(t) = f_{ac}(t) + f_c(t) = f(t)u(-t) + f(t)u(t)$, is*

$$\mathcal{L}[f(t)] = \mathcal{L}[f_{ac}(t)u(t)]_{(-s)} + \mathcal{L}[f_c(t)u(t)] \qquad \mathcal{R}_c \bigcap \mathcal{R}_{ac}$$

One-sided Laplace Transform

One-sided Laplace transform:

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)u(t)e^{-st}dt$$

$f(t)$ either causal or made causal by multiplying it by $u(t)$.

Example Find the Laplace transform of $\delta(t)$, $u(t)$ and a pulse $p(t) = u(t) - u(t-1)$. Use MATLAB also

- $\delta(t)$:

$$\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-s0}dt = \int_{-\infty}^{\infty} \delta(t)dt \qquad \text{ROC: whole } s\text{-plane}$$

- $u(t)$:

$$U(s) = \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \int_{0}^{\infty} e^{-\sigma t}e^{-j\Omega t}dt$$

If $\sigma > 0$, integral converges as $e^{-\sigma t}, t \geq 0$ decays, then

$$U(s) = \frac{e^{-st}}{-s}\Big|_{t=0}^{\infty} = \frac{1}{s} \qquad \text{ROC: } \{(\sigma, \Omega) : \sigma > 0, -\infty < \Omega < \infty\}$$

- $p(t) = u(t) - u(t-1)$, finite support signal so its ROC: whole s-plane

$$P(s) = \mathcal{L}[u(t+1) - u(t-1)] = \int_{-1}^{1} e^{-st} dt = \frac{-e^{-st}}{s}|_{t=-1}^{1} = \frac{1}{s}[e^{s} - e^{-s}] = \frac{e^{s}}{s}[1 - e^{-2s}]$$

$$P(s) = \prod_{k=-\infty, k\neq 0}^{\infty} (s - j\pi k)$$

```
%%%%%%%%%%%%%%%%%
% Example 3.2
%%%%%%%%%%%%%%%%%
syms t s
% Unit-step function
u=sym('heaviside(t)')
U=laplace(u)
% Delta function
d=sym('dirac(t)')
D=laplace(d)
```

giving

```
u = heaviside(t)
U = 1/s

d = dirac(t)
D = 1
```

Example Find and use the Laplace transform of $e^{j(\Omega_0 t+\theta)}u(t)$ to obtain the Laplace transform of $x(t) = \cos(\Omega_0 t + \theta)u(t)$ for $\theta = 0$, $\theta = -\pi/2$. Find the Laplace transform of $\sin(\Omega_0 t)u(t)$. Consider the ROCs.

$$\begin{aligned}
\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] &= \int_0^\infty e^{j(\Omega_0 t+\theta)} e^{-st} dt = e^{j\theta} \int_0^\infty e^{-(s-j\Omega_0)t} dt \\
&= \frac{-e^{j\theta}}{s - j\Omega_0} e^{-\sigma t - j(\Omega-\Omega_0)t} \,|_{t=0}^{\infty} = \frac{e^{j\theta}}{s - j\Omega_0} \qquad \text{ROC:} \;\; \sigma > 0
\end{aligned}$$

Euler's identity:

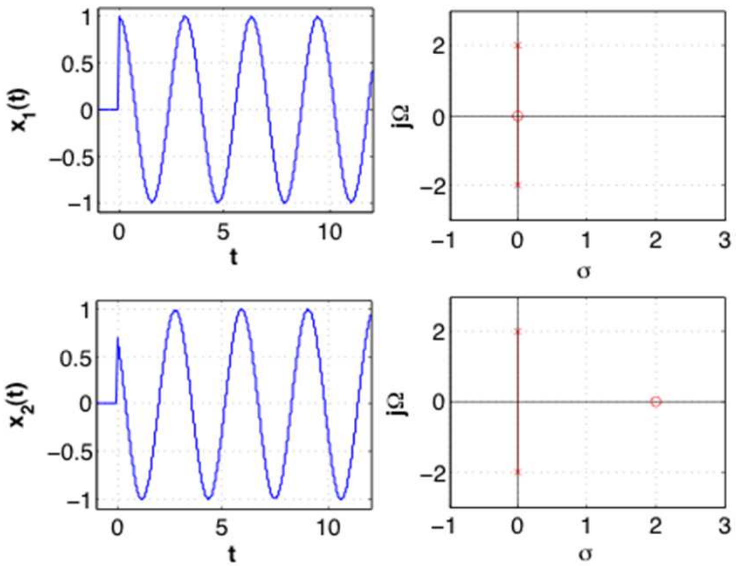
$$\cos(\Omega_0 t + \theta) = \frac{e^{j(\Omega_0 t+\theta)} + e^{-j(\Omega_0 t+\theta)}}{2}$$

linearity of the integral:

$$\begin{aligned}
\mathcal{L}[\cos(\Omega_0 t + \theta)u(t)] &= 0.5\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] + 0.5\mathcal{L}[e^{-j(\Omega_0 t+\theta)}u(t)] = 0.5\frac{e^{j\theta}(s + j\Omega_0) + e^{-j\theta}(s - j\Omega_0)}{s^2 + \Omega_0^2} \\
&= \frac{\cos(\theta)s - \sin(\theta)\Omega_0}{s^2 + \Omega_0^2} \qquad \text{ROC :} \quad \sigma > 0
\end{aligned}$$

Then

$$\begin{aligned}
\theta = 0 \qquad &\Rightarrow \qquad \mathcal{L}[\cos(\Omega_0 t)u(t)] = \frac{s}{s^2 + \Omega_0^2} \\
\theta = -\pi/2 \qquad &\Rightarrow \qquad \mathcal{L}[\sin(\Omega_0 t)u(t)] = \frac{\Omega_0}{s^2 + \Omega_0^2}
\end{aligned}$$



Location of the poles and zeros of $\cos(2t+\theta)u(t)$ *for* $\theta = 0$ *(top figure) and for* $\theta = \pi/4$. *Note that the zero is moved to the right to* 2 *(the value of the frequency) because the zero of the Laplace transform is* $s = \Omega_0 \tan(\theta) = 2\tan(\pi/4) = 2$.

Example $c(t) = e^{-a|t|}$, find its Laplace transform. Determine if it would be possible to compute $|C(\Omega)|^2$

$$\begin{aligned} c(t) &= c(t)u(t) + c(t)u(-t) \\ &= c_c(t) + c_{ac}(t) \end{aligned}$$

$c_c(t)$: causal component, $c_{ac}(t)$ anti-causal components of $c(t)$

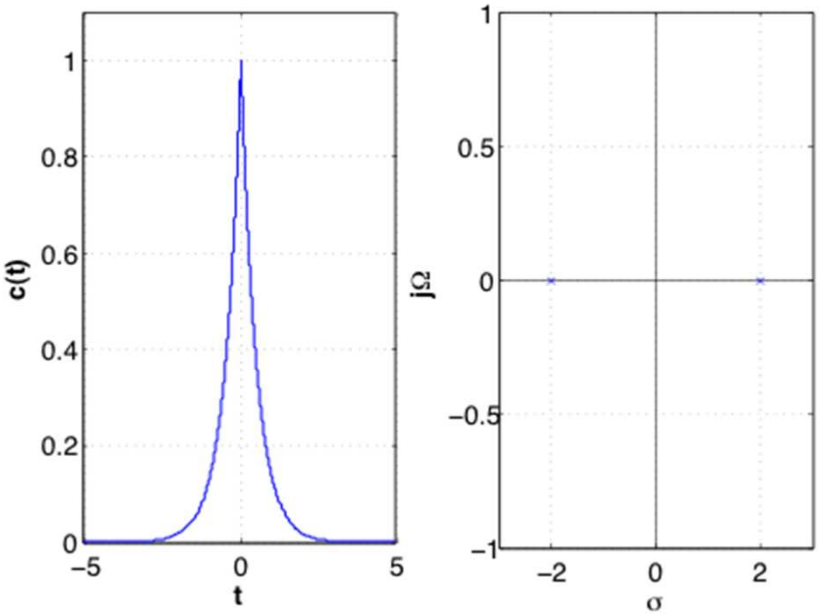
$$C(s) = \mathcal{L}[c_c(t)u(t)] + \mathcal{L}[c_{ac}(-t)u(t)]_{(-s)}$$

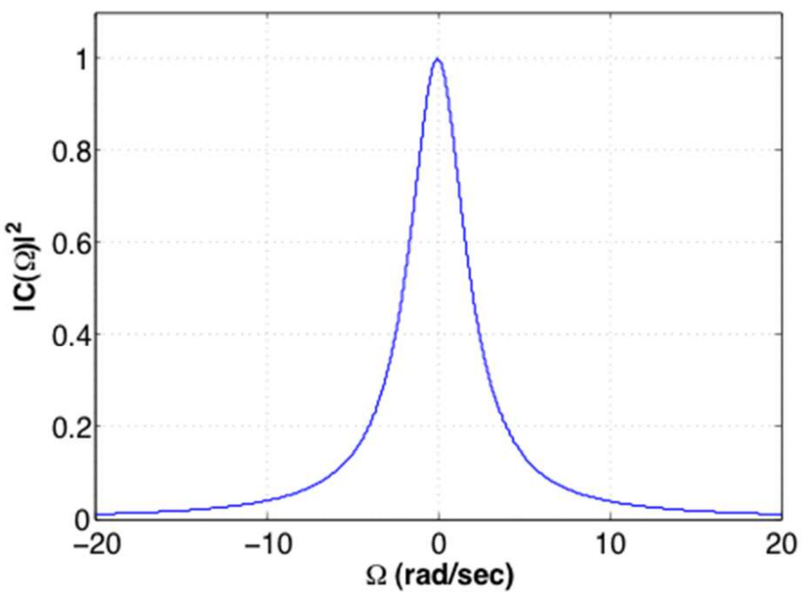
$$C_c(s) = \frac{1}{s+a} \qquad \text{ROC}: \quad \sigma > -a$$

$$\mathcal{L}[c_{ac}(-t)u(t)]_{(-s)} = \frac{1}{-s+a} \qquad \text{ROC:} \quad \sigma < a$$

$$\begin{aligned} C(s) &= \frac{1}{s+a} + \frac{1}{-s+a} \\ &= \frac{2a}{a^2 - s^2} \qquad \text{ROC:} \quad -a < \sigma < a \end{aligned}$$

ROC contains $j\Omega$-axis so $|C(j\Omega)|^2$ can be obtained





Example Laplace transform of a triangular pulse $\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$.

$$R(s) = \int_0^\infty te^{-st}dt = \frac{e^{-st}}{s^2}(-st-1)\,|_{t=0}^{\infty} = \frac{1}{s^2} \qquad \text{ROC:} \;\; \sigma > 0$$

$$\begin{aligned} \frac{d\,U(s)}{ds} &= \int_0^\infty \frac{de^{-st}}{ds}dt \\ &= \int_0^\infty (-t)e^{-st}dt \\ &= -R(s) \end{aligned}$$

then

$$\begin{aligned} R(s) &= -\frac{d\,U(s)}{ds} = \frac{1}{s^2} \\ \Lambda(s) &= \frac{1}{s^2}[e^s - 2 + e^{-s}] \end{aligned}$$

Zeros of $\Lambda(s)$:

s making $e^s - 2 + e^{-s} = (1 - e^{-s})^2 = 0$ or

double $s_k = j2\pi k \qquad k = 0, \pm 1, \pm 2, \cdots$

Basic Properties of One-sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t) \quad \Leftrightarrow \quad \alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t) \quad \Leftrightarrow \quad \alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha) \quad \Leftrightarrow \quad e^{-\alpha s} F(s)$
Frequency shifting	$e^{\alpha t} f(t) \quad \Leftrightarrow \quad F(s-\alpha)$
Multiplication by t	$t\ f(t) \quad \Leftrightarrow \quad sF(s) - f(0-)$
Second derivative	$\dfrac{d^2 f(t)}{dt^2} \quad \Leftrightarrow \quad s^2 F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\displaystyle\int_{0-}^{t} f(t')dt' \quad \Leftrightarrow \quad \frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0 \quad \Leftrightarrow \quad \dfrac{1}{\lvert\alpha\rvert} F\left(\dfrac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Location of the poles (and to some degree the zeros) determines the characteristics of the signal
Signals are characterized by their damping and frequency and as such can be described by the poles of its Laplace transform.

$$f(t) = Ae^{-at}u(t) \qquad \Leftrightarrow \qquad F(s) = \frac{A}{s+a} \qquad \text{ROC: } \sigma > -a$$

σ−axis of s-plane corresponds to damping
a single pole on this axis, in the left-hand s-plane corresponds to a decaying exponential
a single pole on this axis and in the right-hand s-plane corresponds to a growing exponential.

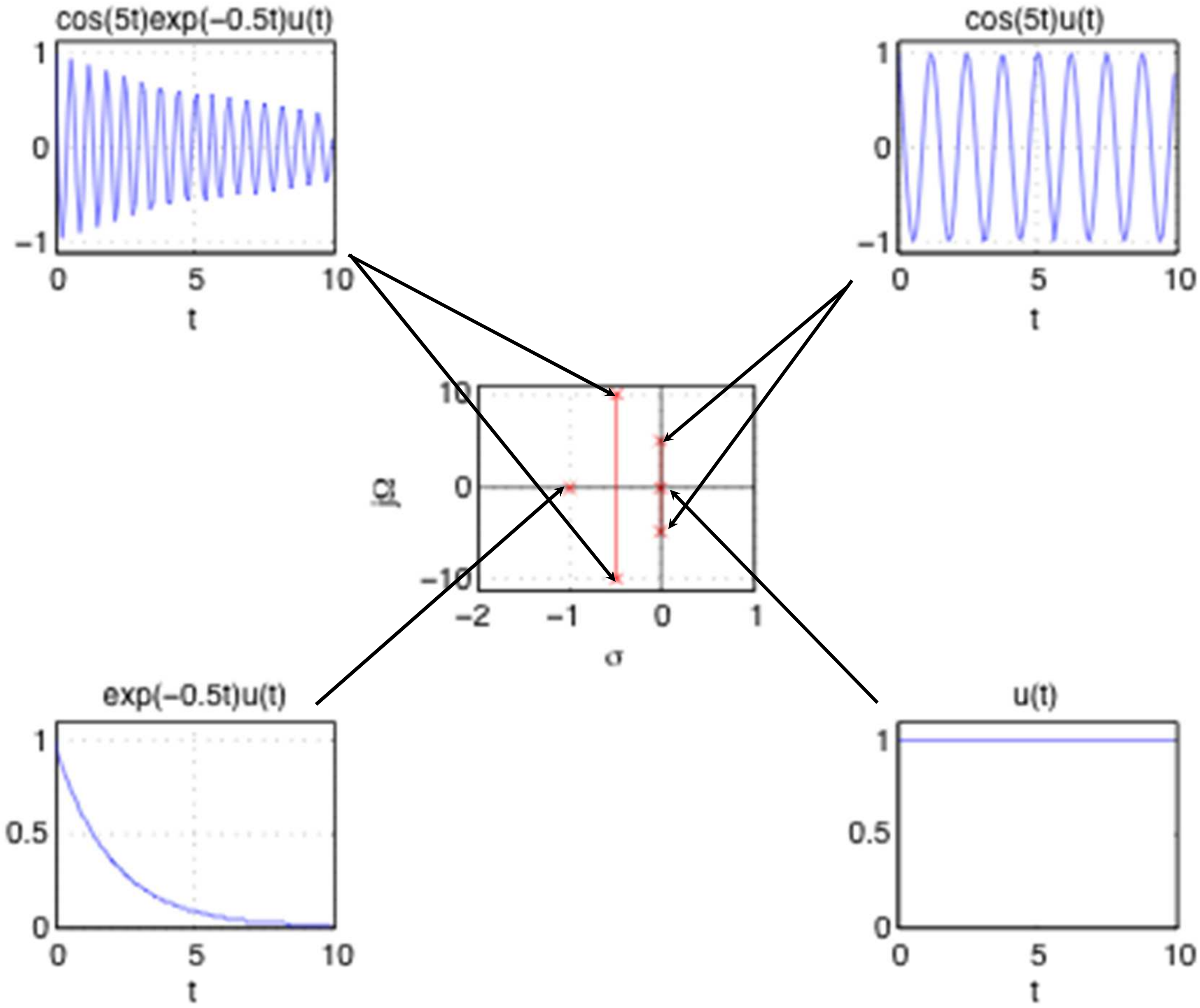
$$g(t) = A\cos(\Omega_0 t)u(t) = 0.5[Ae^{at}u(t) + Ae^{-at}u(t)] \qquad a = j\Omega_0$$

$$G(s) = \frac{A}{2}\frac{1}{s - j\Omega_0} + \frac{A}{2}\frac{1}{s + j\Omega_0} = \frac{As}{s^2 + \Omega_0^2}$$

A sinusoid has complex conjugate pair of poles on the $j\Omega$-axis, requiring negative as well as positive values of the frequency
Moving these poles away from the origin of the $j\Omega$-axis, the frequency increases.

One-sided Laplace Transforms

	Function of time		Function of s, ROC
(1)	$\delta(t)$	$\Leftrightarrow$	$1, \quad \text{whole s-plane}$
(2)	$u(t)$	$\Leftrightarrow$	$\frac{1}{s}, \quad \mathcal{R}e[s] > 0$
(3)	$r(t)$	$\Leftrightarrow$	$\frac{1}{s^2}, \quad \mathcal{R}e[s] > 0$
(4)	$e^{-at}u(t), \; a > 0$	$\Leftrightarrow$	$\frac{1}{s+a}, \quad \mathcal{R}e[s] > -a$
(5)	$\cos(\Omega_0 t)u(t)$	$\Leftrightarrow$	$\frac{s}{s^2+\Omega_0^2}, \quad \mathcal{R}e[s] > 0$
(6)	$\sin(\Omega_0 t)u(t)$	$\Leftrightarrow$	$\frac{\Omega_0}{s^2+\Omega_0^2}, \quad \mathcal{R}e[s] > 0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t), \; a > 0$	$\Leftrightarrow$	$\frac{s+a}{(s+a)^2+\Omega_0^2}, \quad \mathcal{R}e[s] > -a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t), \; a > 0$	$\Leftrightarrow$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}, \quad \mathcal{R}e[s] > -a$
(9)	$2A\, e^{-at}\cos(\Omega_0 t+\theta)u(t), \; a > 0$	$\Leftrightarrow$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle -\theta}{s+a+j\Omega_0}, \quad \mathcal{R}e[s] > -a$
(10)	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\Leftrightarrow$	$\frac{1}{s^N} \; N \text{ an integer}, \quad \mathcal{R}e[s] > 0$
(11)	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\Leftrightarrow$	$\frac{1}{(s+a)^N} \; N \text{ an integer}, \quad \mathcal{R}e[s] > -a$
(12)	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t+\theta)u(t)$	$\Leftrightarrow$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle -\theta}{(s+a+j\Omega_0)^N}, \quad \mathcal{R}e[s] > -a$



cos(5t)exp(−0.5t)u(t)
cos(5t)u(t)
exp(−0.5t)u(t)
u(t)
t
jω
σ

Derivative

$f(t) \quad \Leftrightarrow \quad F(s)$

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \frac{df(t)}{dt}|_{t=0-}$$

If $f^{(N)}(t)$ denotes Nth-order derivative of a function $f(t)$

$$\mathcal{L}[f^{(N)}(t)u(t)] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k}$$

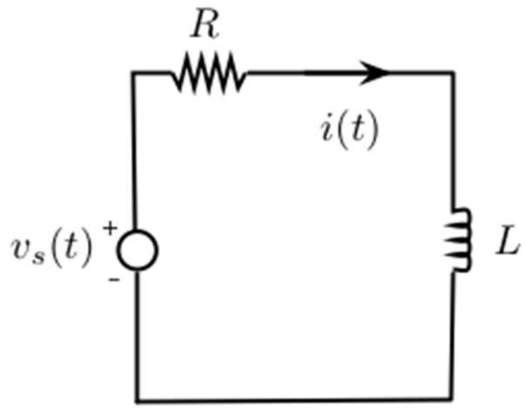
Example Impulse response, $h(t)$, of RL circuit where $i(t)$ is output and $v_s(t)$ the input.
Let $v_s(t) = \delta(t)$ and IC=0

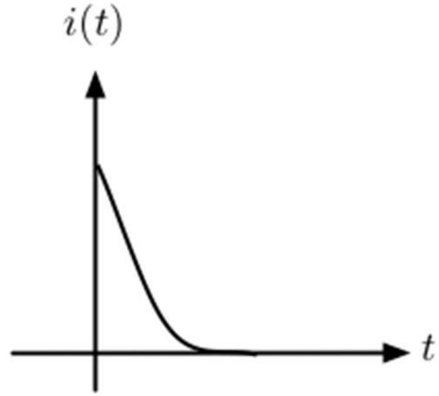
$$v_s(t) = L\frac{di(t)}{dt} + Ri(t) \qquad i(0-) = 0$$

Laplace transform:

$$\begin{aligned} \mathcal{L}[\delta(t)] &= \mathcal{L}[L\frac{di(t)}{dt} + Ri(t)] \\ 1 &= sLI(s) + RI(s) \end{aligned}$$

where $I(s)$ is the Laplace transform of $i(t)$.





Solving for $I(s)$

$$I(s) = \frac{1/L}{s + R/L}$$

so that

$$i(t) = \frac{1}{L} e^{-(R/L)t} u(t).$$

$i(0-) = 0$, response has form of a decaying exponential trying to follow the input signal, a delta function.

Example Duality between the time and the Laplace domains
Connection of $\delta(t)$, $u(t)$ and $r(t)$
Connection with multiple poles
$\delta(t)$, $u(t)$ and $r(t)$:

$$\mathcal{L}[r(t)] = \frac{1}{s^2}$$
$$\mathcal{L}\left[u(t) = \frac{dr(t)}{dt}\right] = s\frac{1}{s^2} = \frac{1}{s}$$
$$\mathcal{L}\left[\delta(t) = \frac{du(t)}{dt}\right] = s\frac{1}{s} = 1$$

In general

$$\frac{d^N X(s)}{ds^N} = \int_0^\infty x(t)\frac{d^N e^{-st}}{ds^N}dt = \int_0^\infty x(t)(-t)^N e^{-st}dt$$

If $x(t) = u(t) \quad \Leftrightarrow \quad X(s) = 1/s$, then $-tx(t) \quad \Leftrightarrow \quad dX(s)/ds = -1/s^2$, or $tu(t) \quad \Leftrightarrow \quad 1/s^2$

Time Shifting

If $\mathcal{L}f(t)u(t) = F(s)$, then Laplace transform of the time-shifted signal $f(t-\tau)u(t-\tau)$ is

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s)$$

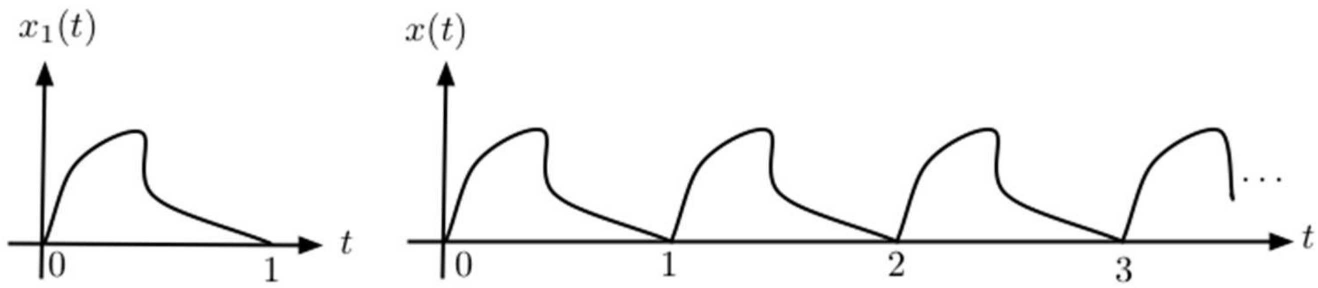
Example Find Laplace transform $x(t)$ where $x_1(t)$ is first pulse, i.e., for $0 \leq t < 1$.

$$x(t) = [x_1(t) + x_1(t-1) + x_1(t-2) + \cdots]u(t)$$

$$\begin{aligned} X(s) &= X_1(s)\left[1 + e^{-s} + e^{-2s} + \cdots\right] \\ &= X_1(s)\left[\frac{1}{1-e^{-s}}\right] \end{aligned}$$

Notice $1 + e^{-s} + e^{-2s} + \cdots = 1/(1-e^{-s})$, cross-mutiplying:

$$[1 + e^{-s} + e^{-2s} + \cdots](1-e^{-s}) = (1 + e^{-s} + e^{-2s} + \cdots) - (e^{-s} + e^{-2s} + \cdots) = 1$$

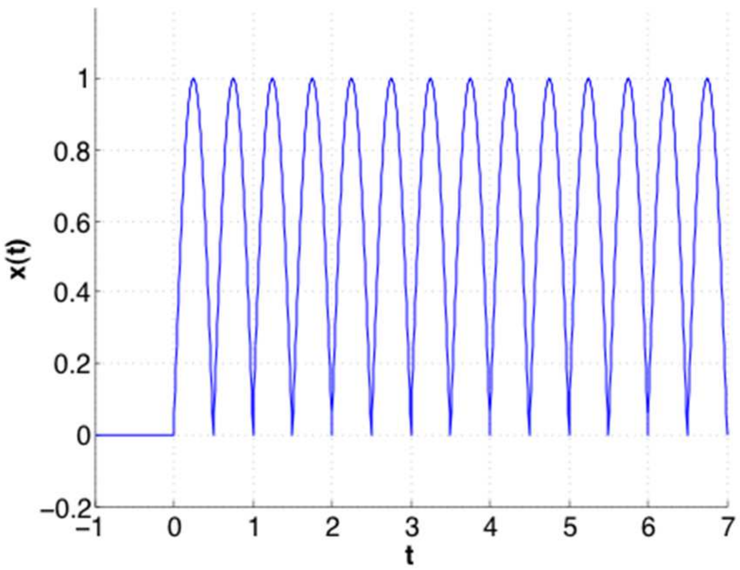


Example Find Laplace transform of causal full-wave rectified signal
First period

$$x_1(t) = \sin(2\pi t)u(t) + \sin(2\pi(t-0.5))u(t-0.5)$$
$$X_1(s) = \frac{2\pi(1+e^{-0.5s})}{s^2+(2\pi)^2}$$

Train of pulses

$$x(t) = \sum_{k=0}^{\infty} x_1(t-0.5k)$$
$$X(s) = X_1(s)[1+e^{-s/2}+e^{-s}+\cdots] = X_1(s)\frac{1}{1-e^{-s/2}} = \frac{2\pi(1+e^{-s/2})}{(1-e^{-s/2})(s^2+4\pi^2)}$$



Convolution integral

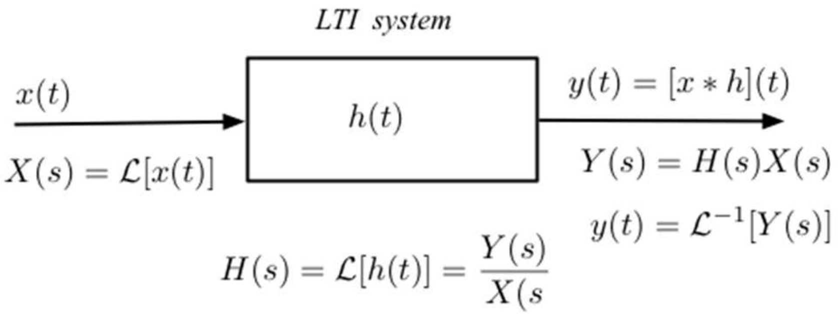
The Laplace transform of the convolution integral of a causal signal $x(t)$, with Laplace transforms $X(s)$, and a causal impulse response $h(t)$, with Laplace transform $H(s)$, is given by

$$\mathcal{L}[(x * h)(t)] = X(s)H(s)$$

The **system function** *or* **transfer function** $H(s) = \mathcal{L}[h(t)]$, *the Laplace transform of the impulse response $h(t)$ of a LTI system, can be expressed as the ratio*

$$H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]} = \frac{\mathcal{L}[\text{ output }]}{\mathcal{L}[\text{ input }]}$$

$H(s)$ transfers the Laplace transform of the input to the output. Just as with the Laplace transform of signals, $H(s)$ characterizes a LTI system by means of its poles and zeros. Thus it becomes a very important tool in the analysis and synthesis of systems.



Inverse Laplace Transform

Inverse of One-sided Laplace Transforms — Partial Fraction Expansion

Causal function $x(t)$, $X(s)$ has ROC

$$\{(\sigma, \Omega) : \sigma > \sigma_{max}, -\infty < \Omega < \infty\}$$

σ_{max} is the maximum of the real parts of the poles of $X(s)$

- Proper rational functions: a rational Laplace transform, i.e.,

$$X(s) = \frac{N(s)}{D(s)}, \qquad N(s), D(s) \text{ polynomials in } s \text{ with real-valued coefficients}$$

is **proper rational** if degree $N(s) <$ degree of $D(s)$
If $X(s)$ is not proper rational do long division, i.e.,

$$X(s) = g_0 + g_1 s + \cdots + g_m s^m + \frac{B(s)}{D(s)}$$

$$x(t) = g_0 \delta(t) + g_1 \frac{d\delta(t)}{dt} + \cdots + g_m \frac{d^m \delta(t)}{dt^m} + \mathcal{L}^{-1}\left[\frac{B(s)}{D(s)}\right]$$

- Remember:
  - the poles of $X(s)$ provide the basic characteristics of the signal $x(t)$,
  - if $N(s)$ and $D(s)$ are polynomials in s with real coefficients, then the zeros and poles of $X(s)$ are real and/or complex conjugate pairs, and can be simple or multiple, and
  - in the inverse, $u(t)$ should be included since the the result of the inverse is causal —the function $u(t)$ is an integral part of the inverse.
- Basic idea of PFE: to decompose proper rational functions into a sum of rational components whose inverse transform can be found directly in tables.

Simple Real Poles

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \qquad \textit{proper rational function}$$

$\{p_k\}$ *simple real poles of* $X(s)$,

$$X(s) = \sum_k \frac{A_k}{s - p_k} \quad \Leftrightarrow \quad x(t) = \sum_k A_k e^{p_k t} u(t)$$

$$A_k = X(s)(s - p_k)\,|_{s=p_k}$$

Example Find causal inverse $x(t)$ of

$$X(s) = \frac{3s+5}{s^2+3s+2} = \frac{3s+5}{(s+1)(s+2)}$$

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Generic answer:

$$x(t) = [A_1 e^{-t} + A_2 e^{-t}]u(t)$$

$$A_1 = X(s)(s+1)|_{s=-1} = \frac{3s+5}{s+2}|_{s=-1} = 2$$
$$A_2 = X(s)(s+2)|_{s=-2} = \frac{3s+5}{s+1}|_{s=-2} = 1$$

$$\text{PFE}: \quad X(s) = \frac{2}{s+1} + \frac{1}{s+2}$$
$$\text{inverse:} \quad x(t) = [2e^{-t} + e^{-2t}]u(t)$$

Check: use the initial or the final value theorems (Table 3.2)

$$\text{initial value theorem:} \; x(0) = 3, \quad \text{coincides with} \quad \lim_{s\to\infty}\left[sX(s) = \frac{3s^2+5s}{s^2+3s+2}\right] = \lim_{s\to\infty} \frac{3+5/s}{1+3/s+2/s^2} = 3$$
$$\text{final value theorem:} \; \lim_{t\to\infty} x(t) = 0, \quad \text{coincides with} \quad \lim_{s\to 0}\left[sX(s) = \frac{3s^2+5s}{s^2+3s+2}\right] = 0$$

Simple Complex Conjugate Poles

$$X(s) = \frac{N(s)}{(s+\alpha)^2 + \Omega_0^2} = \frac{N(s)}{(s+\alpha-j\Omega_0)(s+\alpha+j\Omega_0)}$$ *proper rational function*

with complex conjugate poles $\{s_{1,2} = -\alpha \pm j\Omega_0\}$

$$X(s) = \frac{A}{s+\alpha-j\Omega_0} + \frac{A^*}{s+\alpha+j\Omega_0}$$

where

$$A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta}$$

Inverse function

$$x(t) = 2|A|e^{-\alpha t}\cos(\Omega_0 t + \theta)u(t)$$

Generic response $x(t) = Ke^{-\alpha t}\cos(\Omega_0 t + \Phi)u(t)$

$$A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta}$$
$$X(s)(s+\alpha+j\Omega_0)|_{s=-\alpha-j\Omega_0} = A^*$$

$$\text{Inverse :} \quad x(t) = \left[Ae^{-(\alpha-j\Omega_0)t} + A^*e^{-(\alpha+j\Omega_0)t}\right]u(t)$$
$$= |A|e^{-\alpha t}(e^{j(\Omega_0 t+\theta)} + e^{-j(\Omega_0 t+\theta)})u(t)$$
$$= 2|A|e^{-\alpha t}\cos(\Omega_0 t + \theta)u(t).$$

Example Find causal signal $x(t)$ with

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{2s+3}{(s+1)^2+3}$$

Poles: $-1 \pm j\sqrt{3} \quad \Rightarrow \quad x(t)$ decaying exponential with damping -1
multiplied by a causal cosine of frequency $\sqrt{3}$

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{a+b(s+1)}{(s+1)^2+3} \text{ so that } 3+2s = (a+b)+bs \quad \rightarrow \quad b=2, a=1$$

$$X(s) = \frac{1}{\sqrt{3}} \quad \frac{\sqrt{3}}{(s+1)^2+3} + 2\frac{s+1}{(s+1)^2+3}$$

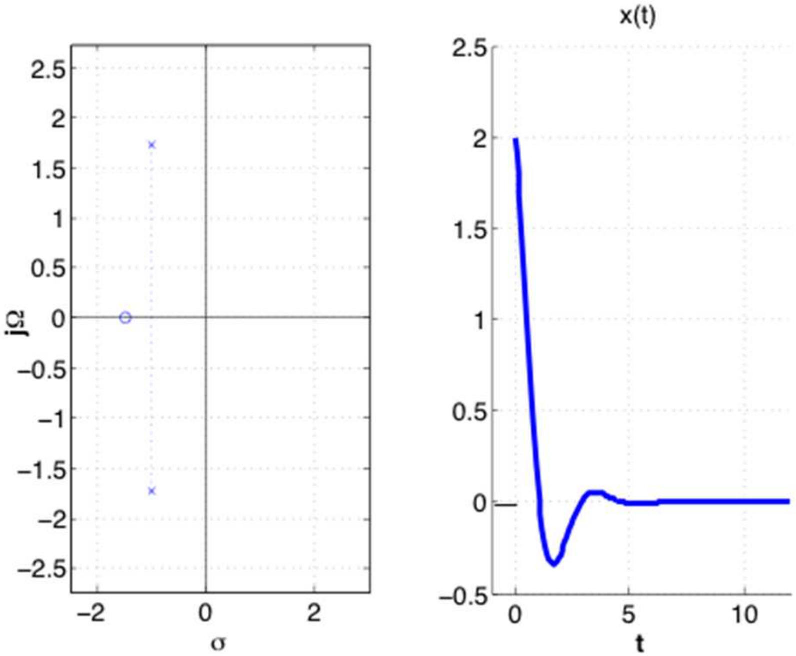
corresponding to

$$x(t) = \left[\frac{1}{\sqrt{3}}\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t)\right] e^{-t}u(t)$$

Initial value theorem:

$$x(0) = 2\text{corresponding to} \lim_{s\to\infty}\left[sX(s) = \frac{2s^2+3s}{s^2+2s+4}\right] = \lim_{s\to\infty}\frac{2+3/s}{1+2/s+4/s^2} = 2$$

```
%%%%%%%%%%%%%%%%%%
% Example 3.15
%%%%%%%%%%%%%%%%%%
clear all; clf
syms s t w
num=[0 2 3]; den=[1 2 4]; % coefficients of numerator and denominator
subplot(121)
splane(num,den) % plotting poles and zeros
disp('>>>>> Inverse Laplace <<<<<')
x=ilaplace((2*s+3)/(s^2+2*s+4)); % inverse Laplace transform
subplot(122)
ezplot(x,[0,12]); title('x(t)')
axis([0 12 -0.5 2.5]); grid
```



Double Real Poles: *Proper rational function has double real poles*

$$X(s) = \frac{N(s)}{(s+\alpha)^2} = \frac{a + b(s+\alpha)}{(s+\alpha)^2} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha}$$

then its inverse is

$$x(t) = [ate^{-\alpha t} + be^{-\alpha t}]u(t)$$

where a *can be computed as*

$$a = X(s)(s+\alpha)^2\,|_{s=-\alpha}$$

and after replacing it, b *is found by computing* $X(s_0)$ *for a value* $s_0 \neq -\alpha$

Example. Find causal $x(t)$ with

$$X(s) = \frac{4}{s(s+2)^2}$$

$$X(s) = \frac{A}{s} + \frac{a + b(s+2)}{(s+2)^2}$$
$$A = X(s)s|_{s=0} = 1$$
$$X(s) - \frac{1}{s} = \frac{-(s+4)}{(s+2)^2} = \frac{a + b(s+2)}{(s+2)^2}$$

Comparing numerators: $b = -1$ and $a + 2b = -4$ or $a = -2$, then we have

$$X(s) = \frac{1}{s} + \frac{-2 - (s+2)}{(s+2)^2}$$

so that

$$x(t) = [1 - 2te^{-2t} - e^{-2t}]u(t)$$

Another way:

$$X(s) = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

Find A and then find B by multiplying both sides by $(s+2)^2$

$$X(s)(s+2)^2|_{s=-2} = \left[\frac{A(s+2)^2}{s} + B + C(s+2)\right]_{s=-2}$$
$$B = X(s)(s+2)^2|_{s=-2}$$

to find C let $s = 1$ we can find the value of C

Example. Find causal $x(t)$ with

$$X(s) = \frac{4}{s((s+1)^2+3)}$$

$$X(s) = \frac{A}{s+1-j\sqrt{3}} + \frac{A^*}{s+1+j\sqrt{3}} + \frac{B}{s}$$
$$B = sX(s)|_{s=0} = 1$$
$$A = X(s)(s+1-j\sqrt{3})|_{s=-1+j\sqrt{3}} = 0.5(-1+\frac{j}{\sqrt{3}}) = \frac{1}{\sqrt{3}}\angle 150^o$$

$$\begin{aligned} x(t) &= \frac{2}{\sqrt{3}} e^{-t}\cos(\sqrt{3}t+150^o)u(t) + u(t) \\ &= -[\cos(\sqrt{3}t) + 0.577\sin(\sqrt{3}t)]e^{-t}u(t) + u(t) \end{aligned}$$

Inverse of Functions Containing $e^{-\rho s}$ Terms

When $X(s)$ has terms $e^{-\rho s}$, ignore them and do PFE on the rest, at the end consider exponential terms

$$X(s) = \frac{N(s)}{D(s)(1-e^{-\alpha s})} = \frac{N(s)}{D(s)} + \frac{N(s)e^{-\alpha s}}{D(s)} + \frac{N(s)e^{-2\alpha s}}{D(s)} + \cdots$$
If $f(t) = \mathcal{L}^{-1}[N(s)/D(s)]$ *then* $x(t) = f(t) + f(t-\alpha) + f(t-2\alpha) + \cdots$

If $X(s) = \frac{N(s)}{D(s)(1+e^{-\alpha s})} = \frac{N(s)}{D(s)} - \frac{N(s)e^{-\alpha s}}{D(s)} + \frac{N(s)e^{-2\alpha s}}{D(s)} - \cdots$
$f(t) = \mathcal{L}^{-1}[N(s)/D(s)]$ *then* $x(t) = f(t) - f(t-\alpha) + f(t-2\alpha) - \cdots$

$$\sum_{k=0}^{\infty} e^{-\alpha s k} = \frac{1}{1-e^{-\alpha s}}$$

verified by cross-multiplying. So when the function is

$$X_1(s) = \frac{N(s)}{D(s)(1-e^{-\alpha s})} = \frac{N(s)}{D(s)} \sum_{k=0}^{\infty} e^{-\alpha s k} = \frac{N(s)}{D(s)} + \frac{N(s)e^{-\alpha s}}{D(s)} + \frac{N(s)e^{-2\alpha s}}{D(s)} + \cdots$$

$f(t) = \mathcal{L}^{-1}[N(s)/D(s)]$ then:

$$x_1(t) = f(t) + f(t-\alpha) + f(t-2\alpha) + \cdots$$

Example Find causal inverse of

$$X(s) = \frac{1-e^{-s}}{(s+1)(1-e^{-2s})}$$

$$X(s) = F(s) \sum_{k=0}^{\infty} (e^{-2s})^k \quad \text{where} \qquad F(s) = \frac{1-e^{-s}}{s+1}$$

$$f(t) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$$

thus

$$x(t) = f(t) + f(t-2) + f(t-4) + \cdots$$

Inverse of Two-sided Laplace Transforms

When finding the inverse of a two-sided Laplace transform

- pay close attention to ROC and location of the poles with respect to the $j\Omega$-axis. Three regions of convergence are possible:
  - a plane to the right of all the poles, corresponding to a causal signal,
  - a plane to the left of all poles, corresponding to an anti-causal signal, and
  - a region that is in between poles on the right and poles on the left (no poles included in it) which corresponds to an anti-causal signal
- If the $j\Omega$-axis is included in the ROC of
  - transfer function $H(s)$, system is BIBO stable and has frequency response
  - $X(s)$ of a signal $x(t)$ then its Fourier transform exists

Example Find the inverse Laplace transform of

$$X(s) = \frac{1}{(s+2)(s-2)} \qquad \text{ROC:} -2 < \mathcal{R}e(s) < 2$$

ROC $-2 < \mathcal{R}e(s) < 2$ equivalent to $\{(\sigma, \Omega) : -2 < \sigma < 2, -\infty < \Omega < \infty\}$

$$X(s) = \frac{1}{(s+2)(s-2)} = \frac{-0.25}{s+2} + \frac{0.25}{s-2} \qquad -2 < \mathcal{R}e(s) < 2$$

- term with pole $s = -2$ corresponds to causal signal, ROC $\mathcal{R}e(s) > -2$,
- term with pole $s = 2$ scorresponds to anti-causal signal, ROC $\mathcal{R}e(s) < 2$

Intersection of ROCs:

$$[\mathcal{R}e(s) > -2] \cap [\mathcal{R}e(s) < 2] = -2 < \mathcal{R}e(s) < 2$$

so that

$$x(t) = -0.25e^{-2t}u(t) - 0.25e^{2t}u(-t)$$

Analysis of LTI systems – Differential Equation representation

Complete response $y(t)$ of system represented by an N^{th}-order linear differential equation

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M$$

$$x(t) \;\; \textit{input}, \;\; y(t) \;\; \textit{output} \;\; \textit{and initial conditions} \{y^{(k)}(t), \;\; 0 \le k \le N-1\}$$

is obtained by inverting the Laplace transform

$$Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s) \qquad Y(s) = \mathcal{L}[y(t)], X(s) = \mathcal{L}[x(t)]$$

$$A(s) = \sum_{k=0}^{N} a_k s^k \qquad a_N = 1$$

$$B(s) = \sum_{\ell=0}^{M} b_\ell s^\ell$$

$$I(s) = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right)$$

$$\textit{Letting} \quad H(s) = \frac{B(s)}{A(s)} \quad \textit{and} \quad H_1(s) = \frac{1}{A(s)}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$
$$Y(s) = H(s)X(s) + H_1(s)I(s)$$

which gives

$$y(t) = y_{zs}(t) + y_{zi}(t)$$
$$\textit{zero-state response} \quad y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$$
$$\textit{zero-input response} \quad y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$$

In terms of convolution integrals

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau + \int_0^t i(\tau)h_1(t-\tau)d\tau$$
$$h(t) = \mathcal{L}^{-1}[H(s)], \ \textit{and} \ h_1(t) = \mathcal{L}^{-1}[H_1(s)]$$
$$i(t) = \mathcal{L}^{-1}[I(s)] = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} y^{(m)}(0)\delta^{(k-m-1)}(t) \right)$$

Example. System represented by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$
$$\text{IC } y(0) = 1,\ dy(t)/dt|_{t=0} = 0,\ \text{input} x(t) = u(t)$$

Find complete response $y(t)$, and impulse response $h(t)$

$$[s^2Y(s) - sy(0) - \frac{dy(t)}{dt}|_{t=0}] + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$
$$Y(s)(s^2 + 3s + 2) - (s + 3) = X(s)$$

after replacing $X(s) = 1/s$

$$Y(s) = \underbrace{\frac{X(s)}{(s+1)(s+2)}}_{\mathcal{L}[y_{zs}(t)]} + \underbrace{\frac{s+3}{(s+1)(s+2)}}_{\mathcal{L}[y_{zi}(t)]} = \frac{1+3s+s^2}{s(s+1)(s+2)} = \frac{B_1}{s} + \frac{B_2}{s+1} + \frac{B_3}{s+2}$$
$$B_1 = 1/2, B_2 = 1, B_3 = -1/2$$

Complete response:

$$y(t) = \underbrace{0.5u(t)}_{\text{steady state}} + \underbrace{(e^{-t} - 0.5e^{-2t})u(t)}_{\text{transient}}$$

To find $H(s) = \mathcal{L}[h(t)]$ from

$$Y(s) = \frac{X(s)}{A(s)} + \frac{I(s)}{A(s)}$$

we need $I(s) = 0$, then $Y(s)/X(s) = H(s) = 1/A(s)$ and $h(t) = e^{-t}u(t) - e^{-2t}u(t)$.

Steady-state and Transient Responses

When solving a d.e. with or without IC:

(i) Steady state response is given by the inverse Laplace transforms of terms of $Y(s)$ *having simple poles (real or complex conjugate pairs) in the* $j\Omega$*-axis,*

(ii) Transient response is given by the inverse transform terms of $Y(s)$ *with poles in the left-hand s-plane, independent of whether the poles are simple or multiple, real or complex*

(iii) Multiple poles in $j\Omega$*-axis and poles in the right-hand s-plane give terms that will increase as t increases.*

Computation of the Convolution Integral

$$Y(s) = \mathcal{L}[y(t) = [x * h](t)] = X(s)H(s)$$
$$X(s) = \mathcal{L}[x(t)], \;\; H(s) = \mathcal{L}[h(t)]$$

Transfer function of the system

$$H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)}$$

$H(s)$ *transfers the Laplace transform* $X(s)$ *of the input into the Laplace transform of the output,* $Y(s)$

Once $Y(s)$ *is found,* $y(t)$ *is computed by means of the inverse Laplace transform*

Example Find convolution $y(t) = [x * h](t)$ when
(i) $x(t) = u(t)$, $h(t) = u(t) - u(t-1)$,
(ii) $x(t) = h(t) = u(t) - u(t-1)$.

- Laplace transforms: $X(s) = \mathcal{L}[u(t)] = 1/s$ and $H(s) = \mathcal{L}[h(t)] = (1 - e^{-s})/s$, so

$$Y(s) = H(s)X(s) = \frac{1 - e^{-s}}{s^2}$$
$$y(t) = r(t) - r(t-1)$$

- Laplace transforms: $X(s) = H(s) = \mathcal{L}[u(t) - u(t-1)] = (1 - e^{-s})/s$ so

$$Y(s) = H(s)X(s) = \frac{(1 - e^{-s})^2}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$
$$y(t) = r(t) - 2r(t-1) + r(t-2)$$

Example RLC circuit: input a voltage source $x(t)$, output the voltage $y(t)$ across the capacitor. Find its impulse response $h(t)$, and its unit-step response $s(t)$. Let $LC = 1$ and $R/L = 2$.
Kirchhoff's voltage law

$$x(t) = Ri(t) + L\frac{di(t)}{dt} + y(t)$$

voltage across the capacitor

$$y(t) = \frac{1}{C}\int_0^t i(\sigma)d\sigma + y(0) \qquad \text{IC}: \ y(0)$$

To obtain d.e. find first and second derivative of $y(t)$:

$$\frac{dy(t)}{dt} = \frac{1}{C}i(t) \quad \Rightarrow \quad i(t) = C\frac{dy(t)}{dt}$$

$$\frac{d^2y(t)}{dt^2} = \frac{1}{C}\frac{di(t)}{dt} \quad \Rightarrow \quad L\frac{di(t)}{dt} = LC\frac{d^2y(t)}{dt^2}$$

$$\text{then} \quad x(t) = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t)$$

a second order differential equation with IC: $y(0)$, initial voltage in the capacitor, and $i(0) = Cdy(t)/dt|_{t=0}$ initial current in the inductor.
Impulse response: $x(t) = \delta(t)$ and IC=0

$$X(s) = [LCs^2 + RCs + 1]Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/LC}{s^2 + (R/L)s + 1/LC} = \frac{1}{(s+1)^2}$$

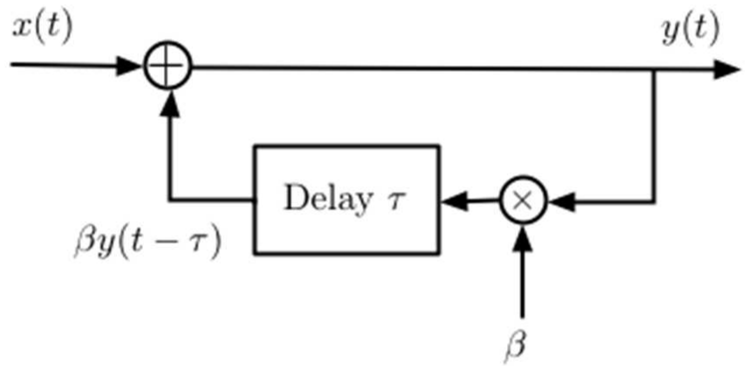
$$h(t) = te^{-t}u(t).$$

Unit–step response: $x(t) = u(t)$, IC=0

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{1}{(s+1)^2}\,\frac{1}{s} = \frac{1}{s} + \frac{-1}{s+1} + \frac{-1}{(s+1)^2} \\ y(t) &= s(t) = u(t) - e^{-t}u(t) - te^{-t}u(t) \end{aligned}$$

$sY(s)$ (derivative of $y(t)$) gives impulse response

Example Positive feedback system: Let $\beta = 1$, $\tau = 1$ and $x(t) = u(t)$, find transfer function, check stability



Input/output equatiom $y(t) = x(t) + \beta y(t-\tau)$
if $x(t) = \delta(t)$, the output $y(t) = h(t)$ or

$$
\begin{aligned}
h(t) &= \delta(t) + \beta h(t-\tau) \\
\text{if} & \quad H(s) = \mathcal{L}[h(t)] \quad \text{then} \\
H(s) &= 1 + \beta H(s) e^{-s\tau} \quad \Rightarrow \quad H(s) = \frac{1}{1-\beta e^{-s\tau}} = \frac{1}{1-e^{-s}} \\
&= \sum_{k=0}^{\infty} e^{-sk} = 1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots
\end{aligned}
$$

Impulse response $h(t)$:

$$h(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \cdots = \sum_{k=0}^{\infty} \delta(t-k)$$

If $x(t)$ is the input,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(\tau-k)x(t-\tau)d\tau \\ &= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \delta(\tau-k)x(t-\tau)d\tau = \sum_{k=0}^{\infty} x(t-k) \end{aligned}$$

replacing $\quad x(t) = u(t) \;\Rightarrow\; y(t) = \sum_{k=0}^{\infty} u(t-k) \to \infty$ as t increases

BIBO stability: $h(t)$ must be absolutely integrable,

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(t-k)dt \\ &= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \delta(t-k)dt = \sum_{k=0}^{\infty} 1 \to \infty \end{aligned}$$

Poles of $H(s)$: roots of $1 - e^{-s} = 0$ or $s_k = \pm j2\pi k$, infinite number of poles on the $j\Omega$-axis, so system is not BIBO stable

What have we accomplished?

- Complex frequency analysis of signals
- Significance of location of poles and zeros
- Convolution integral computation and transfer function
- BIBO stability and pole location
- Solution of differential equations, transient and steady state responses, zero-input and zero-state responses

Where do we go from here?

- Steady-state analysis using Fourier for periodic and aperiodic signals
- Significance of complex exponentials as inputs of LTI systems
- Sinusoidal representation of periodic signals
- Connection of Laplace and Fourier transforms