Signals and Systems Using MATLAB

Luis F. Chaparro

Chapter 2 --- Continuous-time Systems

What is in this chapter?

- Concept of system and system classification
- Linear time-invariant continuous-time systems
  - Linearity and time-invariance
  - Causality and stability
- Systems represented by differential equations
  - Convolution integral
- System interconnection

Continuous-time Systems

System: mathematical transformation of an input signal (or signals) into an output signal (or signals).

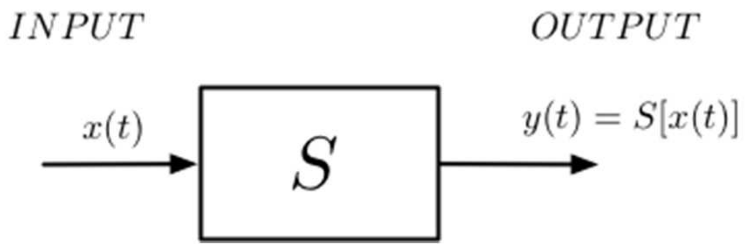
- **Static or dynamic systems**
- **Lumped- or distributed-parameter systems**
- **Passive or active systems**

If input(s) and the output(s) are both continuous-time, discrete-time or digital the corresponding systems are **continuous-time, discrete-time** *or* **digital**, *respectively. It is also possible to have* **hybrid** *systems when the input(s) and the output(s) are not of the same type.*

Linear Time-invariant (LTI) Continuous-time Systems

A continuous-time system is a system in which the signals at its input and output are continuous-time. Mathematically we represent it as a transformation S that converts an input signal $x(t)$ into an output signal $y(t) = S[x(t)]$

$$\begin{array}{ccc} x(t) & \Rightarrow & y(t) = S[x(t)] \\ \textit{Input} & & \textit{Output} \end{array}$$



Characteristics of system model:

- Linearity
- Time-invariance
- Causality
- Stability

Linearity

System S is **linear** *if for inputs $x(t)$ and $v(t)$, and any constants α and β,* **superposition** *holds:*

$$\begin{aligned} S[\alpha x(t) + \beta v(t)] &= S[\alpha x(t)] + S[\beta v(t)] \\ &= \alpha S[x(t)] + \beta S[v(t)] \end{aligned}$$

Example Biased averager: output

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau + B$$

input $x(t)$
System linear?
Solution

$$\begin{aligned} x(t) &\Rightarrow y(t) \\ \alpha x(t) &\Rightarrow \frac{1}{T}\int_{t-T}^{t} \alpha x(\tau)d\tau + B = \frac{\alpha}{T}\int_{t-T}^{t} x(\tau)d\tau + B \end{aligned}$$

not equal to

$$\alpha y(t) = \frac{\alpha}{T}\int_{t-T}^{t} x(\tau)d\tau + \alpha B$$

so system non–linear. System linear if $B = 0$

Example Systems are linear?

$$(i)\ y(t) = |x(t)|$$
$$(ii)\ z(t) = \cos(x(t)) \ \text{ assuming } \ |x(t)| \le 1$$
$$(iii)\ v(t) = x^2(t)$$

$x(t)$, input and $y(t)$, $z(t)$ and $v(t)$ are the outputs

Solution

(i) Superposition not satisfied:

$$\begin{aligned} x_1(t) &\quad\Rightarrow\quad y_1(t) = |x_1(t)| \\ x_2(t) &\quad\Rightarrow\quad y_2(t) = |x_2(t)| \\ x_1(t) + x_2(t) &\quad\Rightarrow\quad y_{12}(t) = |x_1(t) + x_2(t)| \le |x_1(t)| + |x_2(t)| = y_1(t) + y_2(t) \end{aligned}$$

(ii) Nonlinear:

$$\begin{aligned} x(t) &\quad\Rightarrow\quad z(t) = \cos(x(t)) \\ -x(t) &\quad\Rightarrow\quad \cos(-x(t)) = \cos(x(t)) = z(t) \neq -z(t) \end{aligned}$$

(iii) Nonlinear:

$$\begin{aligned} x_1(t) &\quad\Rightarrow\quad v_1(t) = (x_1(t))^2 \\ x_2(t) &\quad\Rightarrow\quad v_2(t) = (x_2(t))^2 \\ x_1(t) + x_2(t) &\quad\Rightarrow\quad (x_1(t) + x_2(t))^2 = (x_1(t))^2 + (x_2(t))^2 + 2x_1(t)x_2(t) \\ &\quad\quad \neq v_1(t) + v_2(t) \end{aligned}$$

<u>Example</u> RLC circuit and linearity

Solution

R voltage-current relation:

$$v(t) = Ri(t)$$

v-i relation is straight line through origin: linear resistor
diode (nonlinear) v-i relation is non linear

C: charge-voltage relation

$$q(t) = Cv_c(t)$$

$$i(t) = dq(t)/dt \quad \Rightarrow \quad i(t) = Cdv_c(t)/dt$$

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0)$$

Capacitor is linear system if $v_c(0) = 0$

$$i_1(t) \qquad \Rightarrow \qquad v_{c1}(t) = \frac{1}{C}\int_0^t i_1(\tau)d\tau$$

$$i_2(t) \qquad \Rightarrow \qquad v_{c2}(t) = \frac{1}{C}\int_0^t i_2(\tau)d\tau$$

$$ai_1(t) + bi_2(t) \qquad \Rightarrow \qquad \frac{1}{C}\int_0^t [ai_1(\tau) + bi_2(\tau)]d\tau = av_{c1}(t) + bv_{c2}(t)$$

L: dual of capacitor, magnetic flux-current relation

$$\phi(t) = Li_L(t)$$

$$v(t) = \frac{d\phi(t)}{dt} = L\frac{di_L(t)}{dt}$$

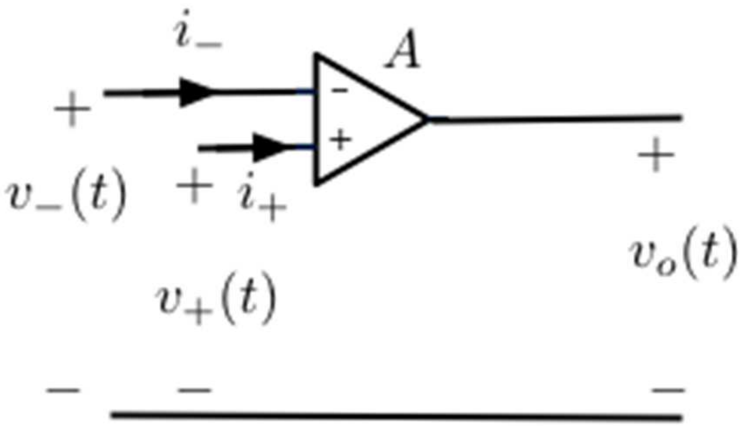
$$i_L(t) = \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0).$$

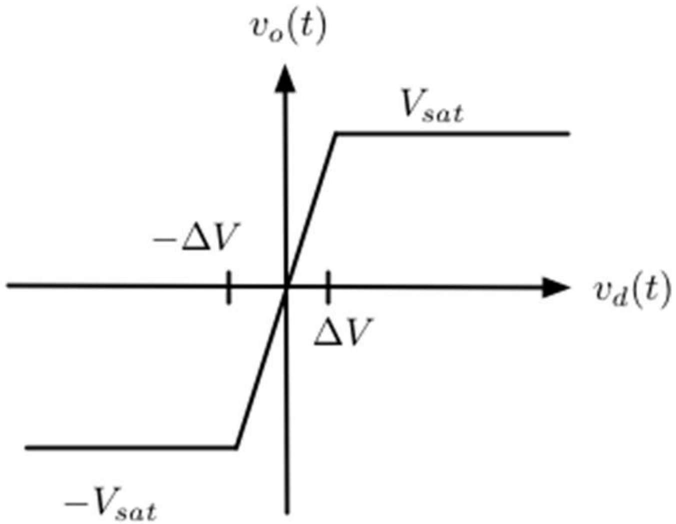
L is linear if $i_L(0) = 0$.

Op-Amp

Inputs voltages:
$v_-(t)$, in *inverting terminal*
$v_+(t)$, in *non-inverting terminal*
Output voltage:

$$v_o(t) = Av_d(t) \qquad -\Delta V \leq v_d(t) \leq \Delta,\ A \text{ very large}$$
$$|v_d(t)| > \Delta V \quad \Rightarrow \quad |v_o(t)| = V_{sat}$$
$$v_d(t) = v_+(t) -_- (t)$$





Linear Op-Amp:

$$A \to \infty$$
$$R_{in} \to \infty$$
virtual short
$$i_- = i_+ = 0$$
$$v_d(t) = v_+(t) - v_-(t) = 0$$

Time-invariance

A continuous-time system S is **time-invariant** *if whenever for an input $x(t)$ with a corresponding output $S[x(t)]$, the output corresponding to a shifted input $x(t \mp \tau)$ (delayed or advanced) is the original output shifted in time $S[x(t \mp \tau)]$ (delayed or advanced). Thus*

$$\begin{aligned} x(t) \quad &\Rightarrow \quad y(t) = S[x(t)] \\ x(t \mp \tau) \quad &\Rightarrow \quad y(t \mp \tau) = S[x(t \pm \tau)] \end{aligned}$$

That is, the system does not age — its parameters are constant
A system that satisfies both the linearity and the time-invariance is called Linear Time-invariant or LTI.

AM/FM Communications Systems

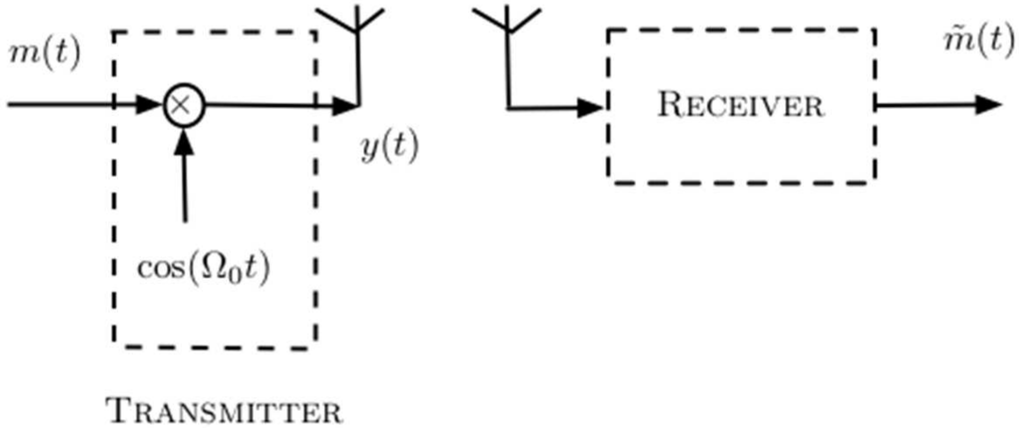
- AM: to radiate"message" over airwaves with reasonably sized antenna

$$\begin{aligned} &\text{voice signal}: \quad 100 \leq f \leq 5KHz \\ &\text{music signal}: \quad 0 \leq f \leq 22KHz \\ &\text{direct transmission requires huge antenna} \end{aligned}$$

Solution: modulation

$$\text{AM} \quad y(t) = m(t)\cos(\Omega_0 t) \quad \text{linear, but time-varying}$$

$$\text{input } m(t-\tau) \quad \Rightarrow \quad \text{output} \quad m(t-\tau)\cos(\Omega_0 t) \neq y(t-\tau) = \cos(\Omega_0(t-\tau))m(t-\tau)$$



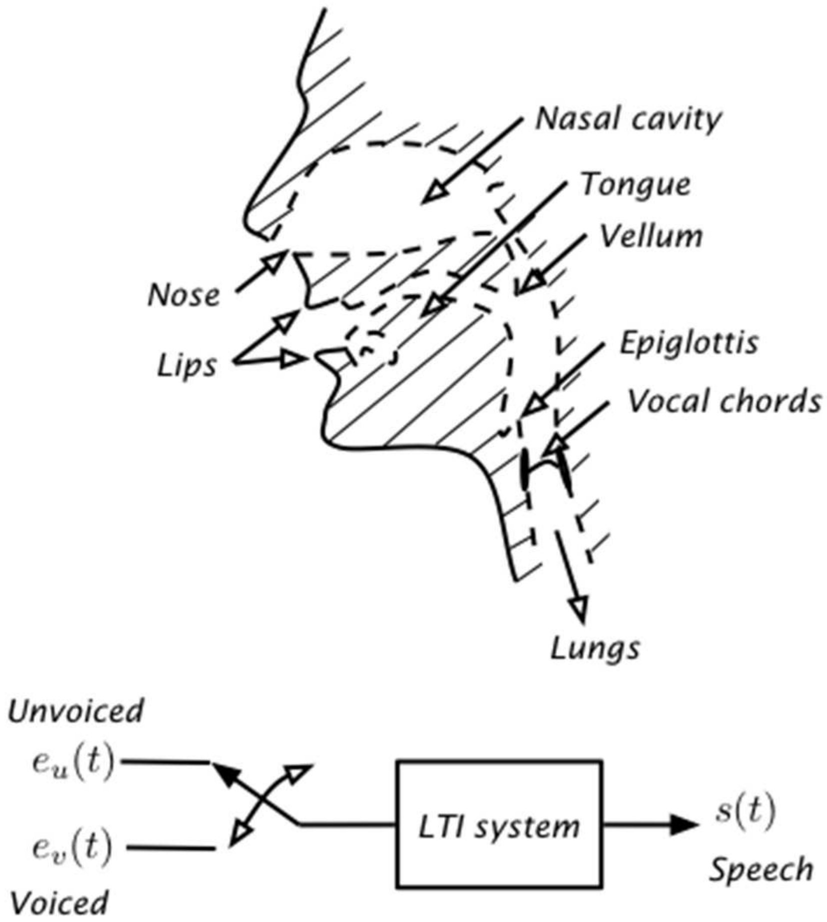
- Frequency modulation (FM) system:

$$z(t) = \cos(\Omega_c t + \int_{-\infty}^{t} m(\tau)d\tau).$$

$m(t)$ is message and $z(t)$ the output
FM is non–linear:

$$\text{scaled message } \gamma m(t) \quad \Rightarrow \quad \text{output} \cos(\Omega_c t + \gamma \int_{-\infty}^{t} m(\tau)d\tau) \neq \gamma z(t)$$

Vocal System



Example Consider time-varying resistors, capacitors and inductors with zero initial conditions in the capacitors and inductors.

Solution

$$v(t) = R(t)i(t)$$

$$q(t) = C(t)v_c(t), \quad i(t) = dq(t)/dt \quad \Rightarrow \quad i(t) = C(t)\frac{dv_c(t)}{dt} + \frac{dC(t)}{dt}v_c(t)$$

$$\phi(t) = L(t)i_L(t), \quad v(t) = d\phi(t)/dt \quad \Rightarrow \quad v(t) = L(t)\frac{di_L(t)}{dt} + \frac{dL(t)}{dt}i_L(t)$$

Example Constant linear capacitors and inductors

$$\frac{dv_c(t)}{dt} = \frac{1}{C} i(t)$$
$$\frac{di_L(t)}{dt} = \frac{1}{L} v(t)$$

initial conditions $v_c(0) = 0$ and $i_L(0) = 0$, under what conditions are these time-invariant systems?

Solution

Consider capacitor (inductor is dual)

$$v_c(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$$

Delay $i(t)$ by λ sec.

$$\frac{1}{C} \int_0^t i(\tau - \lambda) d\tau = \frac{1}{C} \int_{-\lambda}^0 i(\rho) d\rho + \frac{1}{C} \int_0^{t-\lambda} i(\rho) d\rho$$

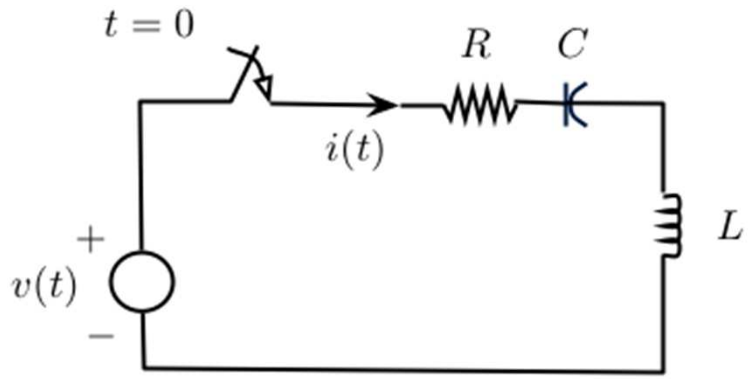
To equal

$$v_c(t - \lambda) = \frac{1}{C} \int_0^{t-\lambda} i(\rho) d\rho,$$

$i(t) = 0$ for $t < 0$

Capacitor time-invariant if $i(t) = 0$ for $t < 0$

If the initial condition $v(0)$ is not zero, or if the input $i(t)$ is not zero for $t < 0$, then linearity or time-invariance, or both are not satisfied.



RLC circuit as LTI system

- Two independent energy storing elements $\Rightarrow$ second-order differential equation with constant coefficients (R, L, C constant)

  $$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau$$

  derivative of $v(t)$ with respect to t:

  $$\frac{dv(t)}{dt} = R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}i(t)$$

  input: $v(t)$, output: $i(t)$

- No initial energy stored in either inductor or capacitor ($i_L(0) = 0$ and $v_C(0) = 0$)
- voltage applied is zero for $t < 0$

Dynamic Systems

A dynamic system represented by a linear differential equation with constant coefficients,

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \qquad t \geq 0$$

N initial conditions: $y(0)$, $d^k y(t)/dt^k|_{t=0}$, for $k = 1, \cdots, N-1$ and input $x(t) = 0$ for $t < 0$, has **complete response** *$y(t)$, for $t \geq 0$ has two components:*

- *the* **zero-state response**, *due exclusively to the input as the initial conditions are zero, and*
- *the* **zero-input response**, *$y_{zi}(t)$, due exclusively to the initial conditions as the input is zero. So that*

$$y(t) = y_{zs}(t) + y_{zi}(t).$$

Thus

1. *When initial conditions are zero, then $y(t)$ depends exclusively on the input (i.e., $y(t) = y_{zs}(t)$), and the system is linear and time-invariant or LTI.*
2. *If initial conditions are different from zero, when checking linearity and time invariance we only change the input and do not change the initial conditions so that $y_{zi}(t)$ remains the same, and thus the system is non–linear. The Laplace transform will provide the solution of these systems.*

Derivative operator

$$D^n[y(t)] = \frac{d^n y(t)}{dt^n} \qquad n > 0, \text{ integer}$$
$$D^0[y(t)] = y(t)$$

Differential equation

$$(a_0 + a_1 D + \cdots + D^N)[y(t)] = (b_0 + b_1 D + \cdots + b_M D^M)[x(t)] \qquad t \geq 0$$
$$D^k[y(t)]_{t=0}, \qquad k = 0, \cdots, N-1$$

Dynamic system LTI if initial conditions and input are zero for $t < 0$, i.e., the system is not energized for $t < 0$.
$x(t)$, ICs two inputs, by superposition
complete solution of d.e. = zero-input solution (due to ICs, $x(t) = 0$)+
zero-state response (due to $x(t)$, ICs=0)
Solve

$$(a_0 + a_1 D + \cdots + D^N)[y(t)] = 0$$
$$\text{ICs} \;\; D^k[y(t)]_{t=0}, \quad k = 0, \cdots, N-1$$

and

$$(a_0 + a_1 D + \cdots + D^N)[y(t)] = (b_0 + b_1 D + \cdots + b_M D^M)[x(t)] \qquad \text{ICs} \;\; D^k[y(t)]_{t=0} = 0 \;\; k = 0, \cdots, N-1$$

Example RL circuit: series connection $R = 1$ and inductor $L = 1$ with voltage source $v(t) = Bu(t)$, and I_0 initial current in L.

- Solve d.e. for $B = 1$ and $B = 2$ and $I_0 = 1$ and $I_0 = 0$
- Zero-input and the zero-output responses?
- LTI?

$$i(t) = [I_0 e^{-t} + B(1 - e^{-t})]u(t)$$

solution of

$$v(t) = i(t) + \frac{di(t)}{dt}$$
$$i(0) = I_0$$

Indeed $i(0+) = I_0$,

$$\underbrace{B}_{v(t)} = \underbrace{[I_0 e^{-t} + B(1 - e^{-t})]}_{i(t)} + \underbrace{[Be^{-t} - I_0 e^{-t}]}_{di(t)/dt} = B \qquad t > 0$$

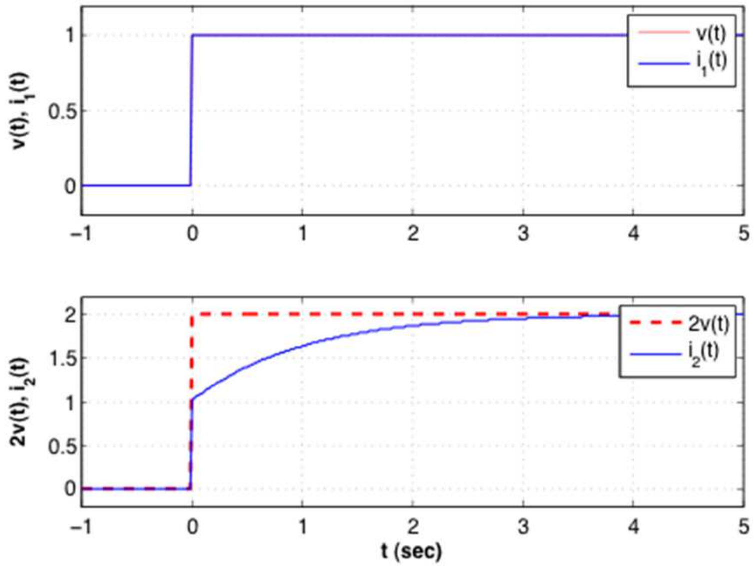
IC $\neq 0$ $I_0 = 1$, $B = 1$, the complete solution

$$i_1(t) = \underbrace{e^{-t}u(t)}_{\text{zero-input resp.}} + \underbrace{(1 - e^{-t})u(t)}_{\text{zero-state resp.}} = u(t)$$

$I_0 = 1$, $B = 2$ (i.e., double original input)

$$i_2(t) = \underbrace{e^{-t}u(t)}_{\text{zero-input resp.}} + \underbrace{2(1 - e^{-t})u(t)}_{\text{zero-state resp.}} = (2 - e^{-t})u(t) \neq 2i_1(t) = 2u(t)$$

System is not linear as zero-input response remains the same (IC did not change)



Non-linear behavior of RL circuit: $I_0 = 1$, $B = 1$, $v(t) = u(t)$, $i_1(t) = u(t)$ *(top)* *and* $I_0 = 1$, $B = 2$, $v(t) = 2u(t)$, $i_2(t) = (2 - e^{-t})u(t)$ *(bottom)*, $i_2(t) \neq 2i_i(t)$.

<u>IC =0</u> $I_0 = 0$, $B = 1$ complete solution

$$i_1(t) = \underbrace{0}_{\text{zero-input resp.}} + \underbrace{(1 - e^{-t})u(t)}_{\text{zero-state resp.}}$$

$I_0 = 0$, $B = 2$ (double the input)

$$\begin{aligned} i_2(t) &= \underbrace{0}_{\text{zero-input resp.}} + \underbrace{2(1 - e^{-t})u(t)}_{\text{zero-state resp.}} \\ &= 2i_1(t) \end{aligned}$$

System is linear (response depends on $v(t)$ only)

Time Invariance $B = 1$, $v(t) = u(t-1)$, I_0, complete response

$$i_3(t) = I_0 e^{-t} u(t) + (1 - e^{-(t-1)}) u(t-1)$$

1. $I_0 = 0$

$$i_3(t) = (1 - e^{-(t-1)}) u(t-1) = i(t-1)$$

   system is time-invariant

2. $I_0 = 1$, complete response not equal to $i(t-1)$ because the term with the initial condition is not shifted like the second term
   System time-varying

3. $I_0 = 0$ system is LTI.

Application of Superposition and Time-invariance

If S is a LTI system, so that

$$y(t) = S[x(t)] \text{ for an input } x(t)$$

then

$$S\left[\sum_k A_k x(t-\tau_k)\right] = \sum_k A_k S[x(t-\tau_k)] = \sum_k A_k y(t-\tau_k)$$

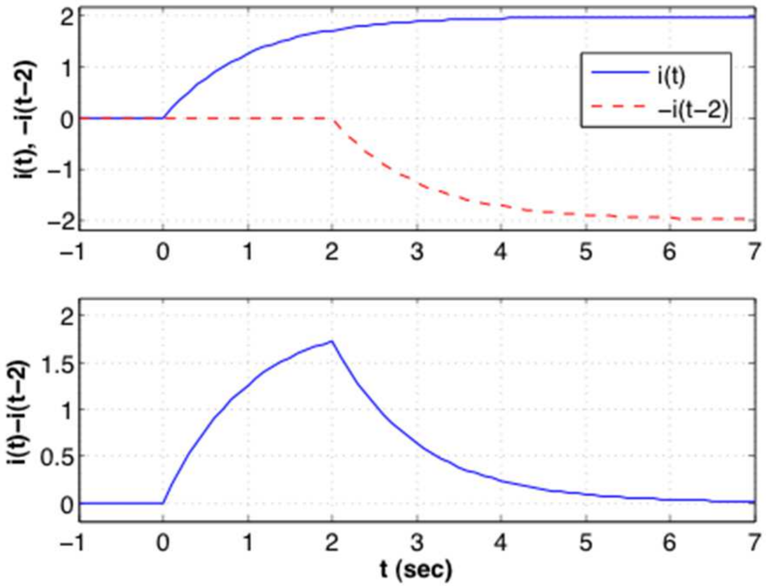
$$S\left[\int g(\tau)x(t-\tau)d\tau\right] = \int g(\tau)S[x(t-\tau)]d\tau = \int g(\tau)y(t-\tau)d\tau$$

Example RL circuit with unit step source:

$$i(t) = (1-e^{-t})u(t)$$

Response to $v(t) = u(t) - u(t-2)$?

$$v(t) = u(t) - u(t-2) \Rightarrow i(t) - i(t-2) = 2(1-e^{-t})u(t) - 2(1-e^{(t-2)})u(t-2)$$



Convolution Integral

Impulse response of LTI system, $h(t)$: output corresponding $\delta(t)$, and IC=0

LTI system S, represented by its impulse response $h(t)$

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \Rightarrow \\ y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= [x*h](t) = [h*x](t) \quad \textit{convolution of} \;\; x(t) \;\; \textit{and} \;\; h(t) \end{aligned}$$

Example Find (i) impulse response of capacitor and (ii) its unit step response.
$C = 1$ F.

C: $v_c(0) = 0$,

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

Impulse response:

$$i(t) = \delta(t) \quad \Rightarrow \quad v_c(t) = h(t) = \frac{1}{C}\int_0^t \delta(\tau)d\tau = \frac{1}{C}u(t)$$

$C = 1$F, unit-step response

$$\begin{aligned} v_c(t) &= \int_{-\infty}^{\infty} h(t-\tau)i(\tau)d\tau = \int_{-\infty}^{\infty} \frac{1}{C}u(t-\tau)u(\tau)d\tau \\ &= \int_0^t d\tau = r(t) \end{aligned}$$

$$h(t) \textbf{ vs } s(t) \textbf{ vs } \rho(t)$$

Impulse response $h(t)$; unit-step response $s(t)$; the ramp response $\rho(t)$ related by

$$h(t) = \begin{cases} ds(t)/dt \\ d^2\rho(t)/dt^2 \end{cases}$$

$$s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau \quad \text{since} \quad u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$$

$$\rho(t) = \int_{-\infty}^{\infty} h(\tau)(t-\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)(t-\tau)d\tau = t\int_{-\infty}^{t} h(\tau)d\tau - \int_{-\infty}^{t} h(\tau)\tau d\tau$$

$$\frac{d\rho(t)}{dt} = \int_{-\infty}^{t} h(\tau)d\tau + th(t) - th(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\frac{d^2\rho(t)}{dt^2} = \frac{d}{dt}\left[\int_{-\infty}^{t} h(\tau)d\tau\right] = h(t)$$

Example Analog averager:

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau \quad \text{input} \quad x(t)$$

accumulation of values of $x(t)$ in a segment $[t-T, t]$ divided by T == average of $x(t)$ in $[t-T, t]$
Find response to ramp

Solution:
$h(t)$: $x(t) = \delta(t)$ and $y(t) = h(t)$ or

$$h(t) = \frac{1}{T}\int_{t-T}^{t} \delta(\tau)d\tau$$

$t < 0$ or $t - T > 0$ integral is zero ($t = 0$, where $\delta(t)$ occurs, not included in integral limits)
$t - T < 0$ and $t > 0$, or $0 < t < T$, integral is 1 ($t = 0$ included in interval)

$$h(t) = \begin{cases} \frac{1}{T} & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

Ramp response $\rho(t)$: input $x(t) = tu(t)$

$$\rho(t) = \frac{1}{T}\int_{t-T}^{t} x(\sigma)d\sigma = \frac{1}{T}\int_{t-T}^{t} \sigma u(\sigma)d\sigma$$

$t - T < 0$, and $t \geq 0$,

$$\rho(t) = \frac{1}{T}\int_{0}^{t} \sigma d\sigma = \frac{t^2}{2T} \qquad 0 \leq t < T$$

$t - T \geq 0$

$$\rho(t) = \frac{1}{T}\int_{t-T}^{t} \sigma d\sigma = \frac{t^2 - (t-T)^2}{2T} = t - \frac{T}{2} \qquad t \geq T$$

$$\rho(t) = \begin{cases} 0 & t < 0 \\ t^2/(2T) & 0 \leq t < T \\ t - T/2 & t \geq T \end{cases}$$

Notice

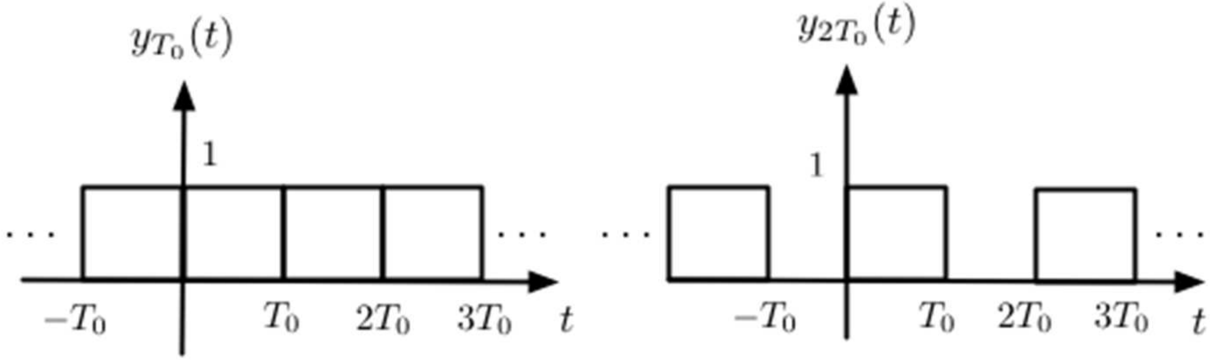
$$\frac{d^2\rho(t)}{dt^2} = \begin{cases} 1/T & 0 \leq t < T \\ 0 & otherwise \end{cases} = h(t)$$

Example Find convolution $y_T(t)$ of pulse $x(t) = u(t) - u(t - T_0)$ with

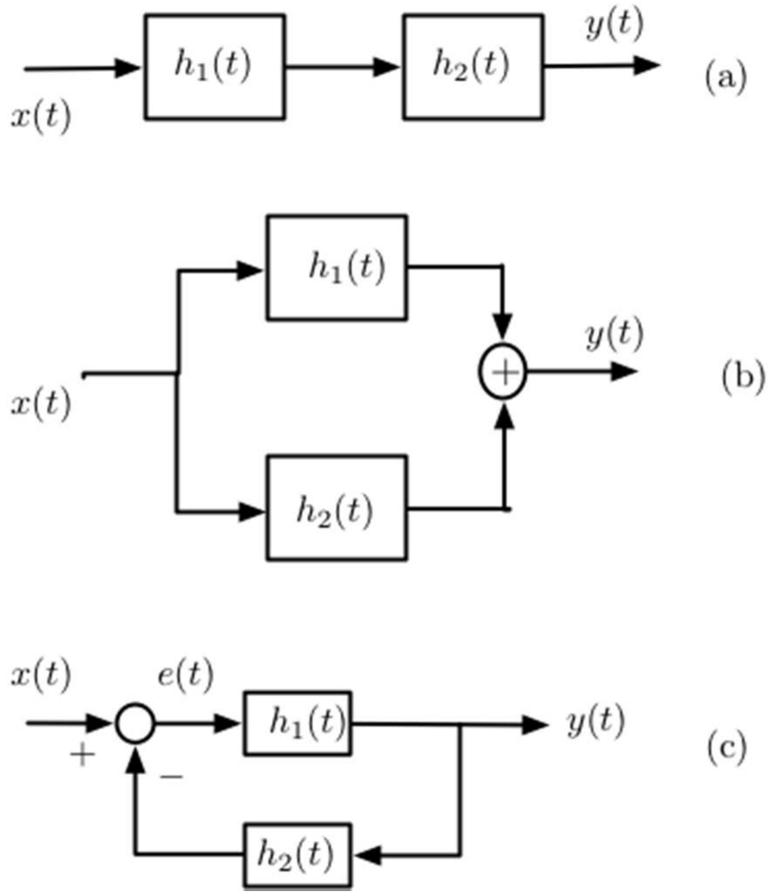
$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$T = T_0$ and $T = 2T_0$, plot corresponding $y_T(t)$

$$\begin{aligned} y_T(t) &= \int_{-\infty}^{\infty} \delta_T(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\tau - kT) x(t-\tau) d\tau \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\tau - kT) x(t-\tau) d\tau = \sum_{k=-\infty}^{\infty} x(t - kT) \int_{-\infty}^{\infty} \delta(\tau - kT) d\tau \\ &= \sum_{k=-\infty}^{\infty} x(t - kT) \quad \text{for any} \quad T \end{aligned}$$



Interconnection of Systems — Block Diagrams



Block diagrams for connecting two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ in (a) cascade, (b) parallel, and (c) negative feedback.

Two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ connected in

(i) Cascade

overall impulse response

$$h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$$

where $h_1(t)$ and $h_2(t)$ commute (i.e., they can be interchanged)

(ii) Parallel

impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t)$$

(iii) Negative Feedback

the output is

$$y(t) = [h_1 * e](t)$$

where the error signal is

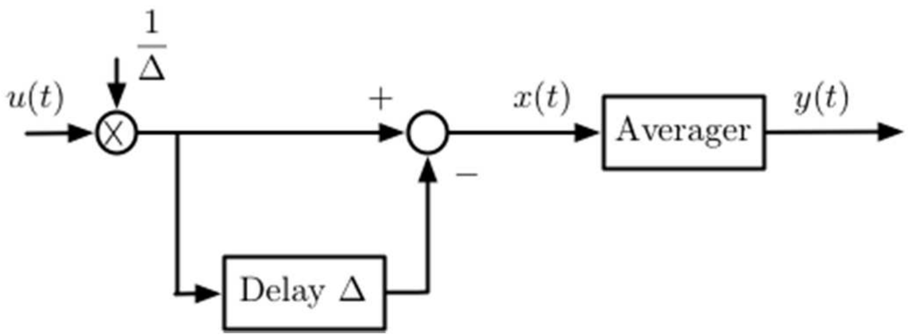
$$e(t) = x(t) - [y * h_2](t)$$

The overall impulse response $h(t)$, or the impulse response of the **closed-loop** *system, is given by the following implicit expression*

$$h(t) = [h_1 - h * h_1 * h_2](t)$$

If $h_2(t) = 0$, i.e., there is no feedback, the system is called **open-loop** *and $h(t) = h_1(t)$.*

Example For block diagram with input $u(t)$. Determine what the system is doing as we let the delay $\Delta \to 0$.



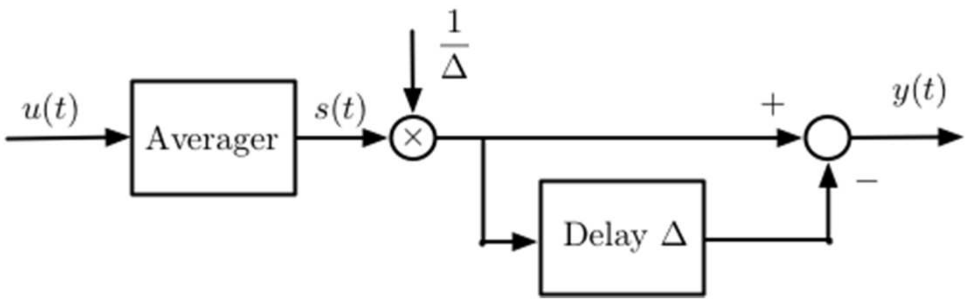
Output of averager is

$$s(t) = \frac{1}{T}\int_{t-T}^{t} u(\tau)d\tau = \begin{cases} 0 & t < 0 \\ t/T & 0 \le t < T \\ 1 & t \ge T \end{cases}$$

$$y(t) = \frac{1}{\Delta}\left[s(t) - s(t-\Delta)\right]$$

If $\Delta \to 0$

$\lim_{\Delta \to 0} y(t) = \frac{ds(t)}{dt} = h(t) = \frac{1}{T}[u(t) - u(t-T)]$ impulse response of the averager



Causality

Continuous-time system S is **causal** *if*

- *for $x(t) = 0$ and no initial conditions, output $y(t) = 0$,*
- *$y(t)$ does not depend on future inputs.*

Non-causal averager

$$
\begin{aligned}
y(t) &= \frac{1}{2T}\int_{t-T}^{t+T} x(\tau)d\tau \\
&= \frac{1}{2T}\int_{t-T}^{t} x(\tau)d\tau + \frac{1}{2T}\int_{t}^{t+T} x(\tau)d\tau.
\end{aligned}
$$

A LTI system represented by impulse response $h(t)$ is **causal** *if*

$$h(t) = 0 \qquad \text{for } t < 0$$

The output of a causal LTI system with a causal input $x(t)$, i.e., $x(t) = 0$ for $t < 0$, is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

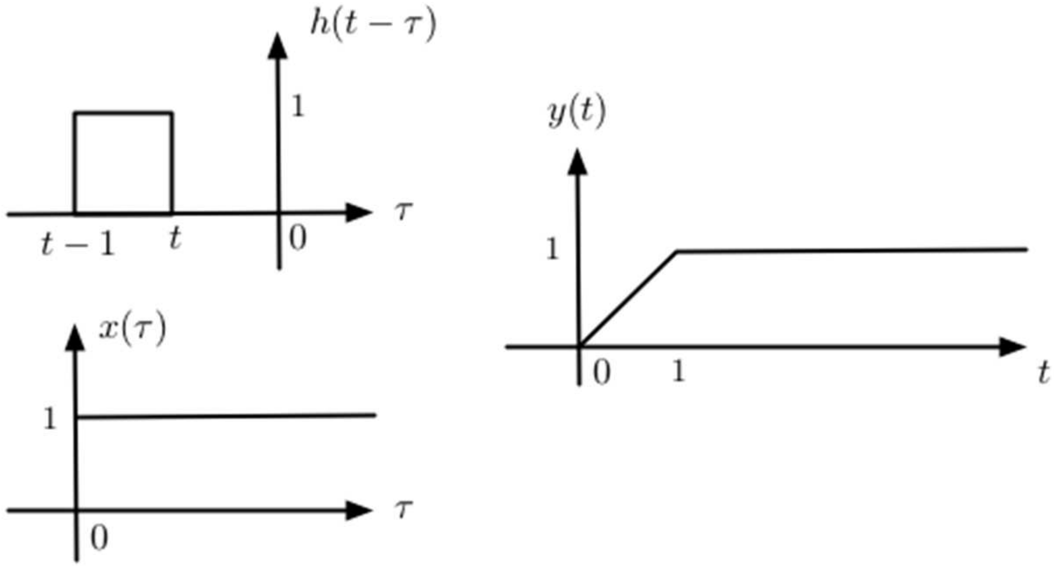
Graphical Computation of Convolution Integral

Graphically, the computation of the convolution integral consists in

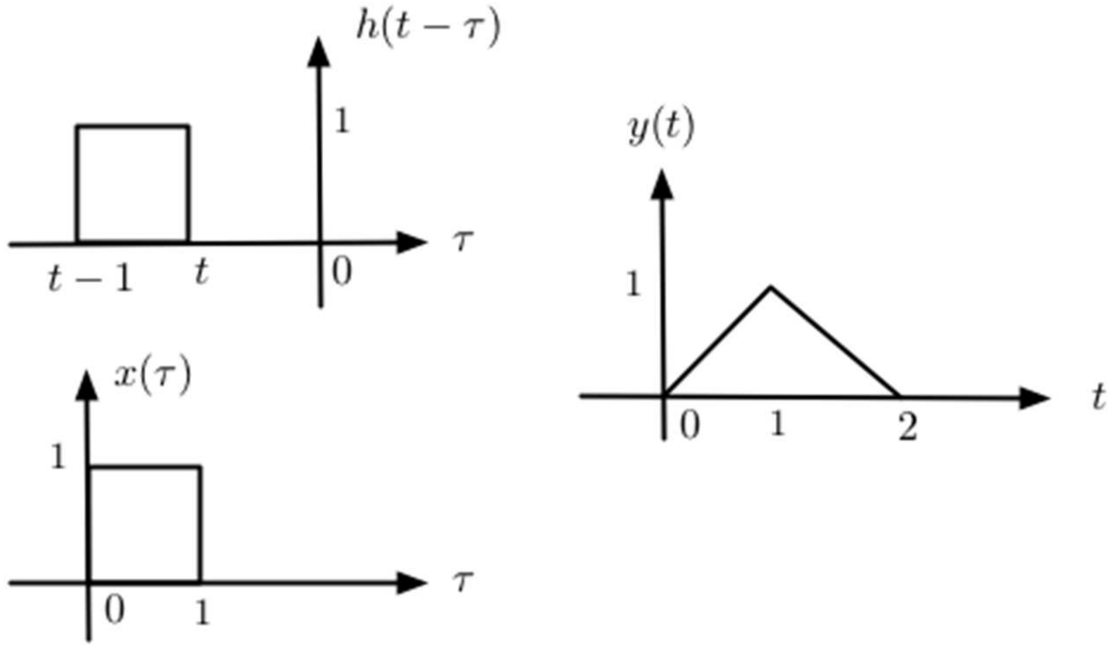
1. multiplying $x(\tau)$ (as a function of τ) by a reflected (again as function of τ) and shifted to the right t sec impulse response $h(t-\tau)$
2. Integrate above product from 0 to t (the time at which we are computing the convolution)
3. Change t and go back to (a). If t final value, stop.

Example Graphically find the unit-step response of an averager $y(t)$ with $T = 1$ sec., with impulse response

$$h(t) = u(t) - u(t-1)$$



Example Graphical computation of the convolution integral of two pulses of the same duration



Bounded-input Bounded-output (BIBO) Stability

BIBO stability: for a bounded (that is what is meant by well-behaved) input $x(t)$ the output of a BIBO stable system $y(t)$ is also bounded. This means that if there is a finite bound $M < \infty$ such that $|x(t)| < M$ (i.e., $x(t)$ in an envelope $[-M, M]$) the output is also bounded.
A LTI system with an absolutely integrable impulse response, i.e.,

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

is BIBO stable.

For a bounded input $|x(t)| < M$, for the output $y(t)$ of a LTI system to be bounded

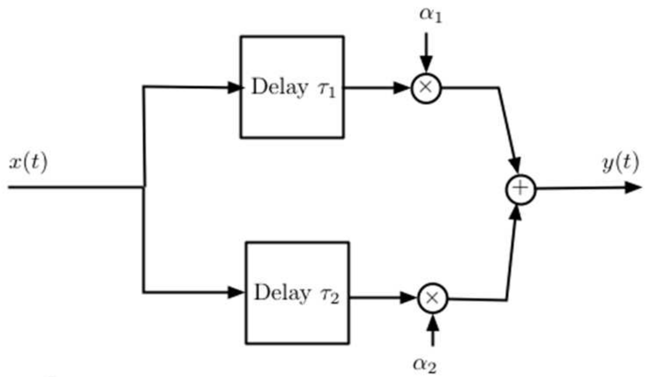
$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \\ &\leq M \int_{-\infty}^{\infty} |h(\tau)|d\tau \\ &\leq ML < \infty \end{aligned}$$

i.e., $h(t)$ is absolutely integrable

Example Causality and BIBO stability of an echo system (or a multi-path system)

$$y(t) = \alpha_1 x(t - \tau_1) + \alpha_2 x(t - \tau_2)$$

$x(t)$ input, and $\alpha_i,\ \tau_i > 0,\ i = 1,\ 2,$ are attenuation factors and delays. Thus the output is the superposition of attenuated and delayed versions of the input. Typically, the attenuation factors are less than unity. Is this system causal and BIBO stable?



Echo system is causal

BIBO stability: $|x(t)| < M < \infty$, for all times

$$|y(t)| \leq |\alpha_1||x(t-\tau_1)| + |\alpha_2||x(t-\tau_2)| < [|\alpha_1| + |\alpha_2|]M$$

output is bounded. The system is BIBO stable.

Alternate way: impulse response, for $x(t) = \delta(t)$

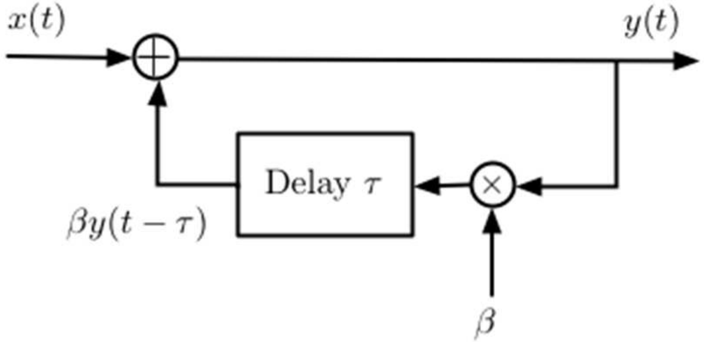
$$y(t) = h(t) = \alpha_1 \delta(t - \tau_1) + \alpha_2 \delta(t - \tau_2)$$

integral

$$\int_{-\infty}^{\infty} |h(t)|dt = |\alpha_1| \int_{-\infty}^{\infty} \delta(t-\tau_1)dt + |\alpha_2| \int_{-\infty}^{\infty} \delta(t-\tau_2)dt = |\alpha_1| + |\alpha_2| < \infty$$

$h(t)$ absolutely integrable, system BIBO stable

Example Positive feedback system: microphone close to a set of speakers that are putting out an amplified acoustic signal. The microphone picks up $x(t)$ and $\beta y(t)$, $|\beta| \geq 1$.
Find equation connecting $x(t)$ and $y(t)$, recursively obtain an expression for $y(t)$ in terms of past values of the input.
Determine if the system is BIBO stable or not, use $x(t) = u(t)$, $\beta = 2$ and $\tau = 1$ in doing so.



$$y(t) = x(t) + \beta y(t - \tau).$$

then

$$y(t - \tau) = x(t - \tau) + \beta y(t - 2\tau)$$

so that

$$y(t) = x(t) + \beta[x(t - \tau) + \beta y(t - 2\tau)] = x(t) + \beta x(t - \tau) + \beta^2 y(t - 2\tau).$$

Repeating

$$y(t) = x(t) + \beta x(t - \tau) + \beta^2 x(t - 2\tau) + \beta^3 x(t - 3\tau) + \cdots$$

If $x(t) = u(t)$ and $\beta = 2$, the output is

$$y(t) = u(t) + 2u(t - 1) + 4u(t - 2) + 8u(t - 3) + \cdots$$

continuously growing as time increases. System is unstable.

What have we accomplished?

- Initiated study of LTI dynamic systems
- Systems represented by differential equations
  - Convolution integral representation
  - Causality and BIBO stability
    - Cascade, parallel and feedback system interconnections

Where do we go from here?

- Frequency analysis (Laplace and Fourier)
  - Develop transfer function system representation
- System response to periodic and aperiodic signals
- Transient, steady-state, zero-input, zero-state responses