Signals and Systems Using MATLAB

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Chapter 1 --- Continuous-time Signals

What is in this chapter?

- Classification of time-dependent signals
- Continuous-time signals
- Basic operations -- even and odd signals
- Periodic signals
- Finite-energy, finite-power signals
- Signal representation using basic signals
- Special signals
- Basic signals operations

Classification of Time-dependent Signals

- *Predictability* : random or deterministic
- *Variation of time and amplitude*: continuous-time, discrete-time, digital
- *Energy*: finite or infinite energy
- *Repetitive behavior*: periodic or aperiodic
- *Symmetry with respect to the time origin*: even or odd
- *Support*: finite or infinite l of the signal outside of which the signal is always zero.

- **Analog to digital converter (ADC)**: converts analog signal into a digital signal
- **digital to analog converter (DAC)**: convert digital signal into an analog signal

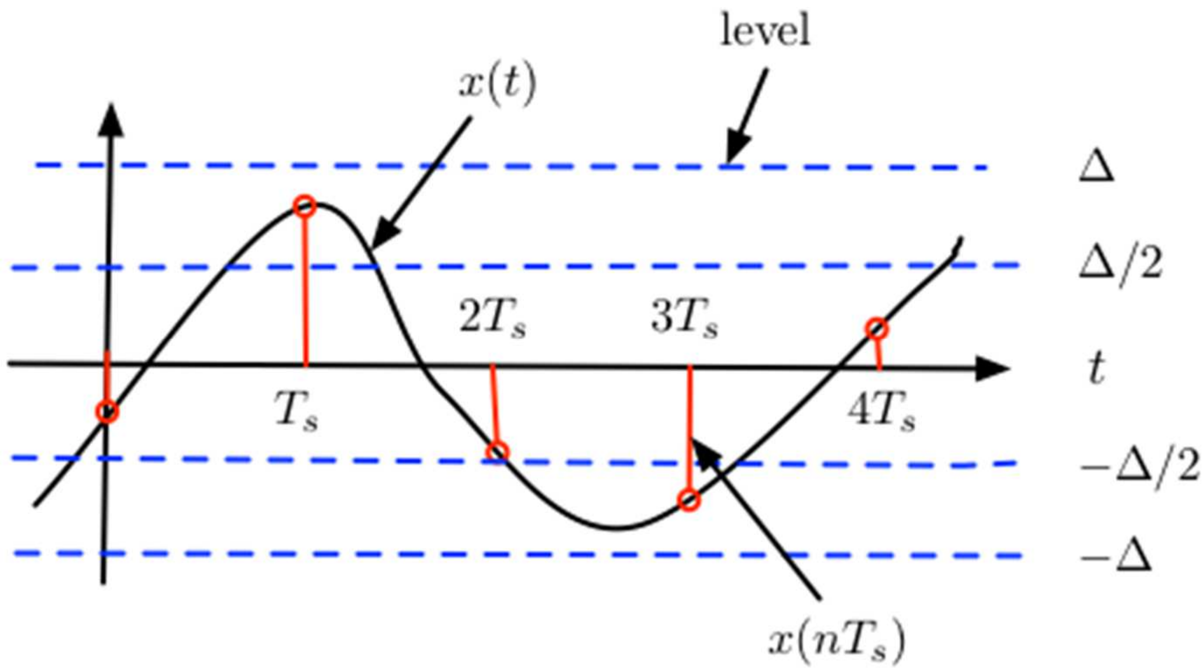


Figure 1. *Discretization in time and in amplitude of an analog signal. The parameters are the sampling period T_s and the quantization level Δ. In time, samples are taken at uniform times $\{nT_s\}$, and in amplitude the range of amplitudes is divided into finite number of levels so that each sample value is approximated by them.*

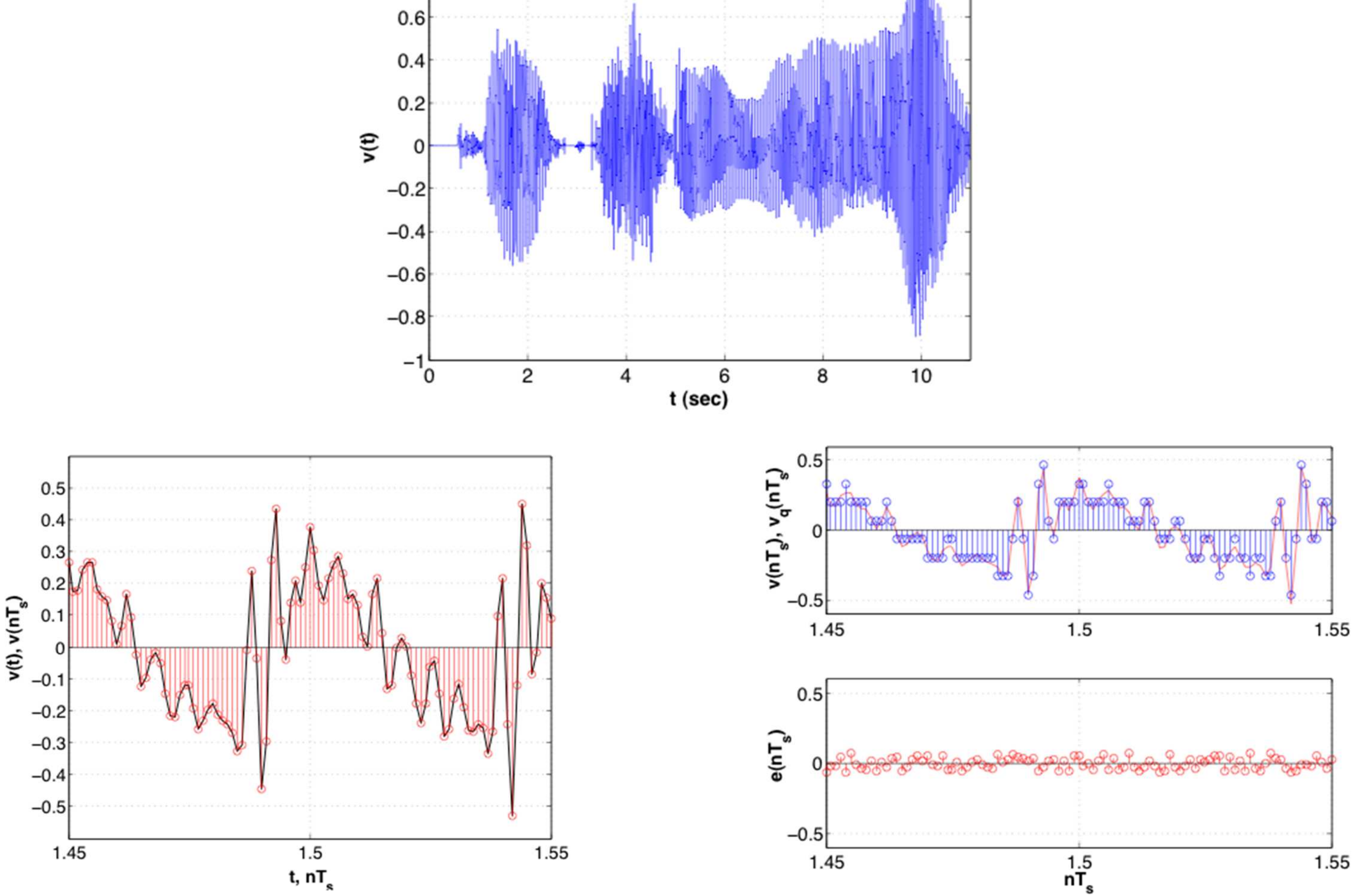


Figure 2. *A segment of the speech signal shown on top is sampled and quantized. The bottom left figure displays the speech segment (continuous line) and the sampled signal (vertical samples) using a sampling period* $T_s = 10^{-3}$ *sec. The bottom-right figure shows the sampled and the quantized signal. The quantization error, difference between the sampled and the quantized signals, is displayed in*

Continuous-time Signals

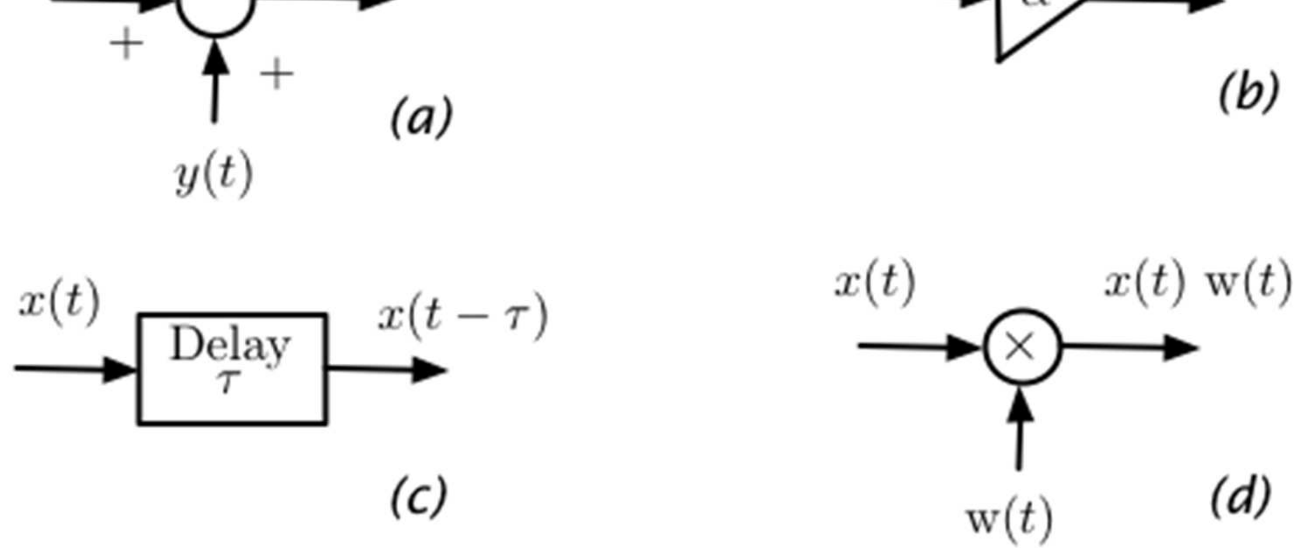
A **continuous-time signal***:*

$$\begin{array}{ccc} x(.) : \mathcal{R} \rightarrow \mathcal{R} & (\mathcal{C}) \\ t & x(t) \end{array}$$

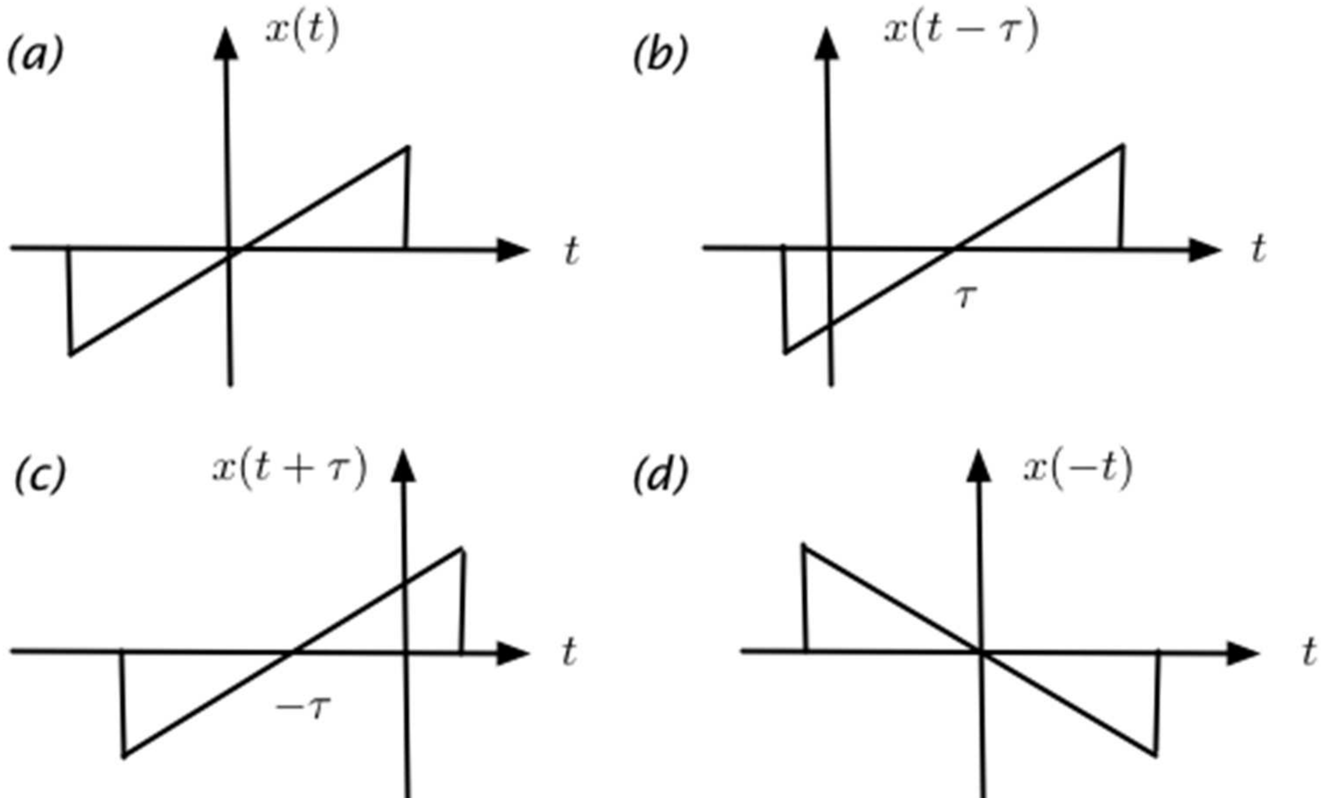
Independent variable is time t
$x(t_0)$, *is a real (or a complex) value*
t *and* $x(t)$ *vary continuously, if needed, from* $-\infty$ *to* ∞.

Basic Signal Operations

- *Signal addition:* $z(t) = x(t) + y(t)$ using adder
- *Constant multiplication* — $z(t) = \alpha x(t)$ using constant multiplier
- *Time/ frequency shifted*:
  - $x(t)$ *delayed* by τ: $x(t-\tau)$
  - $x(t)$ *advanced* by τ: $x(t+\tau)$
  - $x(t)$ *shifted in frequency* or *frequency modulated*: $x(t)e^{j\Omega_0 t}$
- *Time scaled:* $x(\alpha t)$, constant α
  - $\alpha = -1$, $x(-t)$, reversed in time or *reflected*
  - $\alpha \neq 1$, signal compressed/expanded
- *Time windowed*: $z(t) = x(t)\mathrm{w}(t)$, *window signal* $\mathrm{w}(t)$



igure 3. *Diagrams of basic signal operations: (a) adder, (b) constant multiplier, c) delay, and (d) time-windowing or modulation.*



Example Consider an analog pulse

$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $x(t-2)$, $x(t+2)$, and $x(-t)$.

Solution

$$x(t-2) = \begin{cases} 1 & 0 \le t-2 \le 1 \text{ or } 2 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$x(0)$ (in $x(t)$ occurs at $t=0$) in $x(t-2)$ occurs when $t=2$, so signal has been shifted to the right 2, $x(t-2)$ is "delayed" by 2 with respect to $x(t)$

$$x(t+2) = \begin{cases} 1 & 0 \le t+2 \le 1 \text{ or } -2 \le t \le -1 \\ 0 & \text{otherwise} \end{cases}$$

$x(0)$ for $x(t+2)$ occurs at $t=-2$ which is ahead of $t=0$

$$x(-t) = \begin{cases} 1 & 0 \le -t \le 1 \text{ or } -1 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

or mirror image of the original, e.g., $x(1)$ occurs when $t=-1$.

Example Compare $x(-t+2)$ to $x(t)$ above

Solution

$x(-t+2)$ is reflected, but advanced or delayed by 2?

t	$x(-t+2)$
2	$x(0)=1$
1.5	$x(0.5)=1$
1	$x(1)=1$
0	$x(2)=0$
-1	$x(3)=0$

then $x(-t+2)$ reflected and "delayed" by 2

Even and Odd Signals

Even and odd signals are defined as follows:

$$x(t) \quad \textbf{even}: \quad x(t) = x(-t)$$
$$x(t) \quad \textbf{odd}: \quad x(t) = -x(-t)$$

Even and odd decomposition: *Any signal $y(t)$ is representable as a sum of an even component $y_e(t)$ and an odd component $y_o(t)$*

$$y(t) = y_e(t) + y_o(t)$$

where

$$\begin{aligned} y_e(t) &= 0.5\,[y(t) + y(-t)] \\ y_o(t) &= 0.5\,[y(t) - y(-t)] \end{aligned}$$

Example Consider

$$x(t) = \cos(2\pi t + \theta) \qquad -\infty < t < \infty$$

- What θ makes $x(t)$ even, odd?
- For $\theta = \pi/4$ is $x(t)$ even or odd?

Solution

Reflection $x(-t) = \cos(-2\pi t + \theta)$, then:
(i) $x(t)$ even if $x(t) = x(-t)$ or

$$\begin{aligned} \cos(2\pi t + \theta) &= \cos(-2\pi t + \theta) \\ &= \cos(2\pi t - \theta) \end{aligned}$$

or $\theta = -\theta$ or $\theta = 0, \pi$
$x_1(t) = \cos(2\pi t)$ and $x_2(t) = \cos(2\pi t + \pi) = -\cos(2\pi t)$ are even
(ii) $x(t)$ odd if $x(t) = -x(-t)$ or

$$\cos(2\pi t + \theta) = -\cos(-2\pi t + \theta) = \cos(-2\pi t + \theta \pm \pi) = \cos(2\pi t - \theta \mp \pi)$$

or $\theta = -\theta \mp \pi$ or $\theta = \mp\pi/2$
$\cos(2\pi t - \pi/2) = \sin(2\pi t)$ and $\cos(2\pi t + \pi/2) = -\sin(2\pi t)$ are odd
$\theta = \pi/4$, $x(t) = \cos(2\pi t + \pi/4)$ is neither even nor odd

$$x(t) = \begin{cases} 2\cos(4t) & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Find its even and odd decomposition
- What would happen if $x(0) = 2$ instead of 0, i.e., when we define the sinusoid at $t = 0$? Explain.

Solution

$x(t)$ is neither even nor odd because $x(t) = 0$ for $t \leq 0$

$$x_e(t) = 0.5[x(t) + x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ \cos(4t) & t < 0 \\ 0 & t = 0 \end{cases}$$

$$x_o(t) = 0.5[x(t) - x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ -\cos(4t) & t < 0 \\ 0 & t = 0 \end{cases}$$

adding them gives $x(t)$

For$x(0) = 2$,

$$x_e(t) = 0.5[x(t) + x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ \cos(4t) & t < 0 \\ 2 & t = 0 \end{cases}$$

odd component is the same

Periodic and Aperiodic Signals

An analog signal $x(t)$ *is* **periodic** *if*

- *it is defined for all possible values of* t, $-\infty < t < \infty$, *and*
- *there is a positive real value* T_0, *the* **period** *of* $x(t)$, *such that*

$$x(t + kT_0) = x(t)$$

for any integer k.

The period of $x(t)$ *is the smallest possible value of* $T_0 > 0$ *that makes the periodicity possible. Thus, although* NT_0 *for an integer* $N > 1$ *is also a period of* $x(t)$ *it should not be considered the period.*

Example Periodic signal $x(t)$ of period T_0, are following signals periodic? If so their periods

1. $y(t) = A + x(t)$,
2. $z(t) = x(t) + v(t)$, $v(t)$ periodic of period $T_1 = NT_0$, where $N > 0$ integer,
3. $w(t) = x(t) + u(t)$, $u(t)$ periodic of period T_1, not multiple of T_0, conditions for $w(t)$ periodic

Solution
(a) $y(t)$ periodic of period T_0, i.e., integer k, $y(t+kT_0) = A+x(t+kT_0) = A+x(t)$ since $x(t)$ is periodic of period T_0.
(b) $T_1 = NT_0$ period of $x(t)$
$z(t)$ periodic of period T_1, integer k

$$z(t+kT_1) = x(t+kT_1) + v(t+kT_1) = x(t+kNT_0) + v(t) = x(t) + v(t)$$

(d) $w(t)$ periodic if

$$\frac{T_1}{T_0} = \frac{N}{M}$$

where $N > 0$ and $M > 0$ integers not divisible by each other

$$w(t+MT_1) = x(t+MT_1) + u(t+MT_1) = x(t+NT_0) + u(t+MT_1) = x(t) + u(t)$$

- $w(t) = x(t)y(t)$, periodic?
- $p(t) = (1 + x(t))(1 + y(t))$ periodic?

Solution

$$x(t) = \cos(2t) + j\sin(2t) \quad \text{periodic, } T_0 = \pi$$
$$y(t) = \cos(\pi t) + j\sin(\pi t) \quad \text{periodic, } T_1 = 2$$

(1) $z(t)$ periodic if $T_1/T_0 = 2/\pi$ is rational, which is not
(2) $w(t) = x(t)y(t) = e^{j(2+\pi)t} = \cos(\Omega_2 t) + j\sin(\Omega_2 t)$ periodic
$\Omega_2 = 2 + \pi = 2\pi/T_2$ so $T_2 = 2\pi/(2+\pi)$
(3) $1 + x(t)$, periodic, $T_0 = \pi$
$1 + y(t)$, periodic, $T_1 = 2$

$$p(t) = 1 + x(t) + y(t) + x(t)y(t)$$

$x(t) + y(t)$ not periodic, then $p(t)$ is not periodic

1. *Analog sinusoids of frequency $\Omega_0 > 0$ are periodic of period $T_0 = 2\pi/\Omega_0$. If $\Omega_0 = 0$ the period is not well defined.*

2. *The sum of two periodic signals $x(t)$ and $y(t)$, of periods T_1 and T_2, is periodic if the ratio of the periods T_1/T_2 is a rational number N/M, with N and M non-divisible. The period of the sum is $MT_1 = NT_2$.*

3. *The product of two sinusoids is periodic. The product of two periodic signals*

Finite Energy and Finite Power Signals

The **energy** *and the* **power** *of an analog signal* $x(t)$ *are defined for either finite or infinite support signals as:*

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

The signal $x(t)$ *is then said to be* **finite energy**, *or* **square integrable**, *whenever*

$$E_x < \infty$$

The signal is said to have **finite power** *if*

$$P_x < \infty$$

(a) $x(t) = \cos(\pi t/2 + \pi/4), \quad -\infty < t < \infty$,
(b) $y(t) = (1 + j)e^{j\pi t/2}$, $0 \le t \le 10$, zero otherwise,
(c) $z(t) = 1$, for $0 \le t \le 10$ and zero otherwise.
Finite energy, finite power or both?

Solution

Energy

$$E_x = \int_{-\infty}^{\infty} \cos^2(\pi t/2 + \pi/4)dt \to \infty \ \text{ infinite energy}$$

$$E_y = \int_0^{10} |(1+j)e^{j\pi t/2}|^2 dt = 2\int_0^{10} dt = 20 \ \text{ finite energy}$$

$$E_z = \int_0^{10} dt = 10 \ \text{ finite energy}$$

Power:

$$P_y, P_z = 0, \ y(t), z(t) \ \text{ finite energy}$$

For $x(t)$, let $T = NT_0$:

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{2}{2T}\int_0^T \cos^2(\pi t/2 + \pi/4)dt = \lim_{N\to\infty} \frac{1}{NT_0}\int_0^{NT_0} \cos^2(\pi t/2 + \pi/4)dt \\ &= \lim_{N\to\infty} \frac{1}{NT_0}\left[N\int_0^{T_0} \cos^2(\pi t/2 + \pi/4)dt\right] = \frac{1}{T_0}\int_0^{T_0} \cos^2(\pi t/2 + \pi/4)dt \\ &= \frac{1}{8}\int_0^4 \cos(\pi t + \pi/2)dt + \frac{1}{8}\int_0^4 dt = 0 + 0.5 = 0.5 \end{aligned}$$

$$x(t) = \cos(2\pi t) + \cos(4\pi t), \quad -\infty < t < \infty$$

$$y(t) = \cos(2\pi t) + \cos(2t), \quad -\infty < t < \infty$$

Periodic? Power?

Solution

$\cos(2\pi t)$, $\cos(4\pi t)$ periods $T_1 = 1$ and $T_2 = 1/2 \Rightarrow x(t)$ periodic $(T_1/T_2 = 2)$ with period $T_1 = 2T_2 = 1$, and harmonically related frequencies

$\cos(2t)$, period $T_3 = \pi \Rightarrow y(t)$ not periodic ($T_1/T_3 = 1/\pi$ not rational), frequencies 2π and 2 not harmonically related

$$\begin{aligned}
x^2(t) &= \cos^2(2\pi t) + \cos^2(4\pi t) + 2\cos(2\pi t)\cos(4\pi t) \\
&= 1 + \frac{1}{2}\cos(4\pi t) + \frac{1}{2}\cos(8\pi t) + \cos(6\pi t) + \cos(2\pi t) \\
P_x &= \frac{1}{T_0}\int_0^{T_0} x^2(t)dt = 1
\end{aligned}$$

$$\begin{aligned}
y^2(t) &= \cos^2(2\pi t) + \cos^2(2t) + 2\cos(2\pi t)\cos(2t) \\
&= 1 + \frac{1}{2}\cos(4\pi t) + \frac{1}{2}\cos(4t) + \cos(2(\pi+1)t) + \cos(2(\pi-1)t) \\
P_y &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y^2(t)dt \\
&= 1 + \frac{1}{2T_4}\int_0^{T_4}\cos(4\pi t)dt + \frac{1}{2T_5}\int_0^{T_5}\cos(4t)dt \\
&\quad + \frac{1}{T_6}\int_0^{T_6}\cos(2(\pi+1)t)dt + \frac{1}{T_7}\int_0^{T_7}\cos(2(\pi-1)t)dt = 1
\end{aligned}$$

If

$$x(t) = \cos(2\pi t) + \cos(4\pi t) = x_1(t) + x_2(t)$$
$$y(t) = \cos(2\pi t) + \cos(2t) = y_1(t) + y_2(t)$$

then $P_{x_1} = P_{x_2} = P_{y_1} = P_{y_2} = 0.5$ so that

$$P_x = P_{x_1} + P_{x_2} = 1$$
$$P_y = P_{y_1} + P_{y_2} = 1$$

The power of a sum of sinusoids,

$$x(t) = \sum_k A_k \cos(\Omega_k t) = \sum_k x_k(t)$$

with harmonically or non harmonically related frequencies $\{\Omega_k\}$, *is the sum of the power of each of the sinusoidal components,*

$$P_x = \sum_k P_{x_k}$$

Complex Exponentials

A **complex exponential** *is a signal of the form*

$$\begin{aligned} x(t) &= Ae^{at} \\ &= |A|e^{rt}\left[\cos(\Omega_0 t+\theta)+j\sin(\Omega_0 t+\theta)\right] \quad -\infty < t < \infty \end{aligned}$$

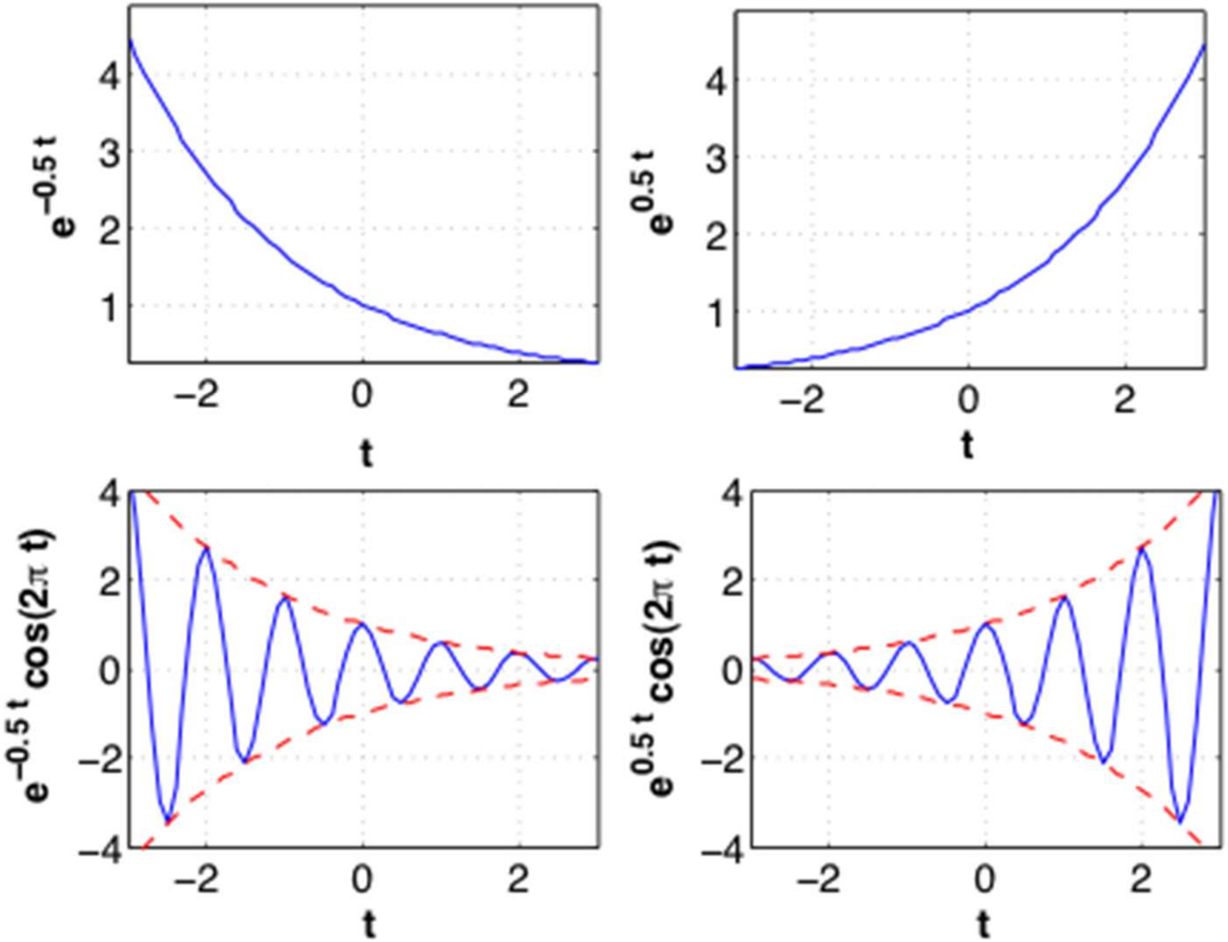
where $A=|A|e^{j\theta}$*, and* $a=r+j\Omega_0$ *are complex numbers.*

A and a are real,

$x(t)=Ae^{at} \qquad -\infty < t < \infty$, decaying exponential $(a<0)$, growing exponential $(a>0)$

A is real, $a=j\Omega_0$,

$$x(t) = Ae^{j\Omega_0 t} = \underbrace{A\cos(\Omega_0 t)}_{\mathcal{R}e[x(t)]} + j\underbrace{A\sin(\Omega_0 t)}_{\mathcal{I}m[x(t)]}$$



Analog exponentials: decaying exponential (top left), growing exponential (top right), modulated exponential decaying and growing (bottom left and right).

Sinusoids are of the general form

$$A\cos(\Omega_0 t+\theta) = A\sin(\Omega_0 t+\theta+\pi/2) \qquad -\infty < t < \infty$$

where A is the amplitude of the sinusoid, $\Omega_0 = 2\pi f_0$ (rad/sec) is the frequency, and θ is a phase shift. The frequency and time variables are inversely related,

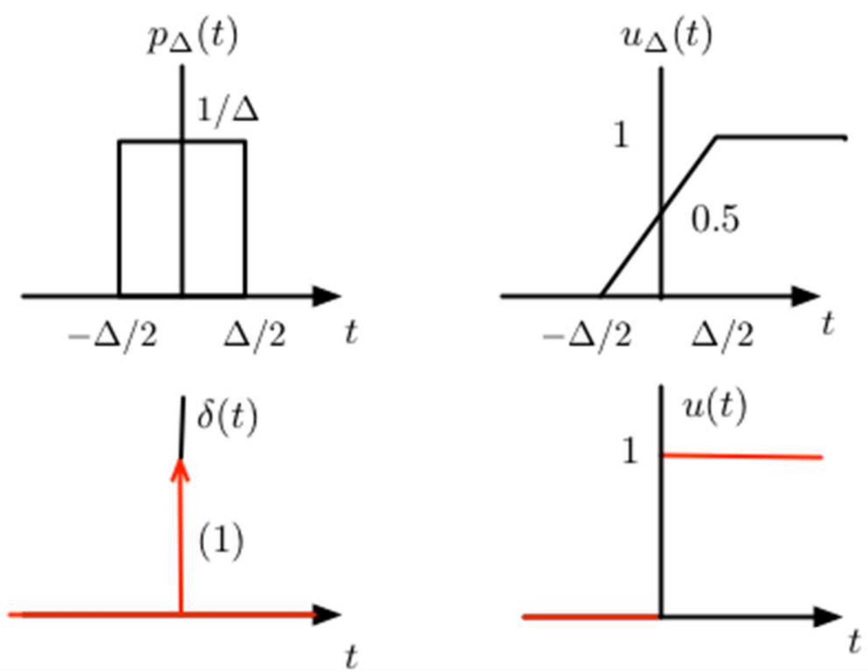
$$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$\begin{aligned}\cos(\Omega_0 t) &= \frac{1}{2}(e^{j\Omega_0 t}+e^{-j\Omega_0 t})\\ \sin(\Omega_0 t) &= \frac{1}{2j}(e^{j\Omega_0 t}-e^{-j\Omega_0 t})\end{aligned}$$

Modulation systems in communications

$$A(t)\cos(\Omega(t)t+\theta(t))$$

- **Amplitude modulation or AM:** $A(t)$ changes according to the message, frequency and phase constant,
- **Frequency/Phase modulation or FM:** $\Omega(t)/\theta(t)$ changes according to



The **impulse signal** $\delta(t)$ *is:*

- *zero everywhere except at the origin where its value is not well defined, i.e.,* $\delta(t) = 0$, $t \neq 0$, *undefined at* $t = 0$,
- *its area is unity, i.e.,*

$$\int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

The **unit-step** *signal is*

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau \\ \delta(t) &= \frac{du(t)}{dt} \end{aligned}$$

Calculus: $\Delta \to 0$, relation between $u(t)$ and $\delta(t)$

$$\begin{aligned} u_\Delta(t) &= \int_{-\infty}^{t} p_\Delta(\tau)d\tau \\ p_\Delta(t) &= \frac{du_\Delta(t)}{dt} \end{aligned}$$

The **ramp** *signal is defined as*

$$r(t) = t\, u(t)$$

Its relation to the unit-step and the unit-impulse signals is

$$\frac{dr(t)}{dt} = u(t)$$
$$\frac{d^2r(t)}{dt^2} = \delta(t)$$

$$x_1(t) = \cos(2\pi t)[u(t) - u(t-1)]$$
$$x_2(t) = u(t) - 2u(t-1) + u(t-2)$$

represent them as the sum of a continuous signal and unit step signals, and find their derivatives.

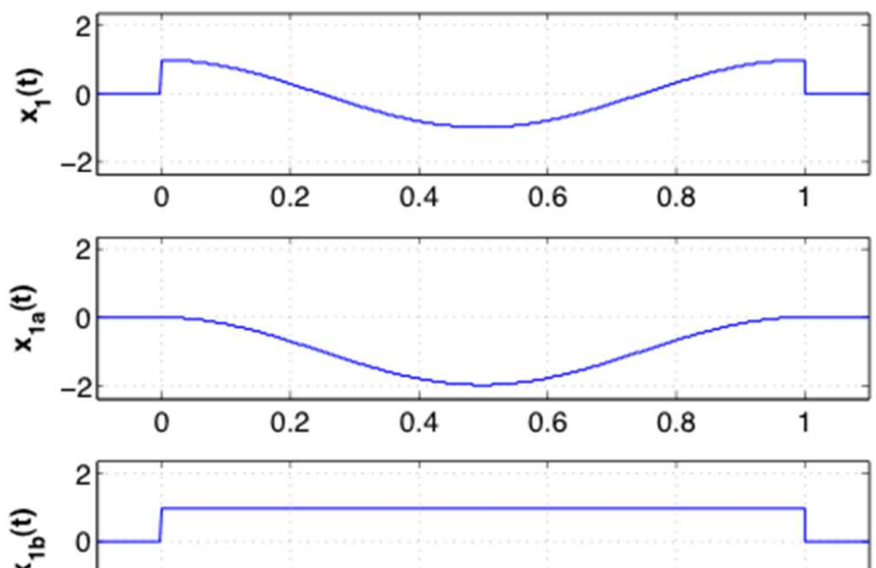
Solution

$$x_1(t) = \underbrace{(\cos(2\pi t) - 1)[u(t) - u(t-1)]}_{continuous} + \underbrace{[u(t) - u(t-1)]}_{discontinuous}$$

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -2\pi \sin(2\pi t)[u(t) - u(t-1)] + (\cos(2\pi t) - 1)[\delta(t) - \delta(t-1)] + \delta(t) - \delta(t-1) \\ &= -2\pi \sin(2\pi t)[u(t) - u(t-1)] + \delta(t) - \delta(t-1) \end{aligned}$$

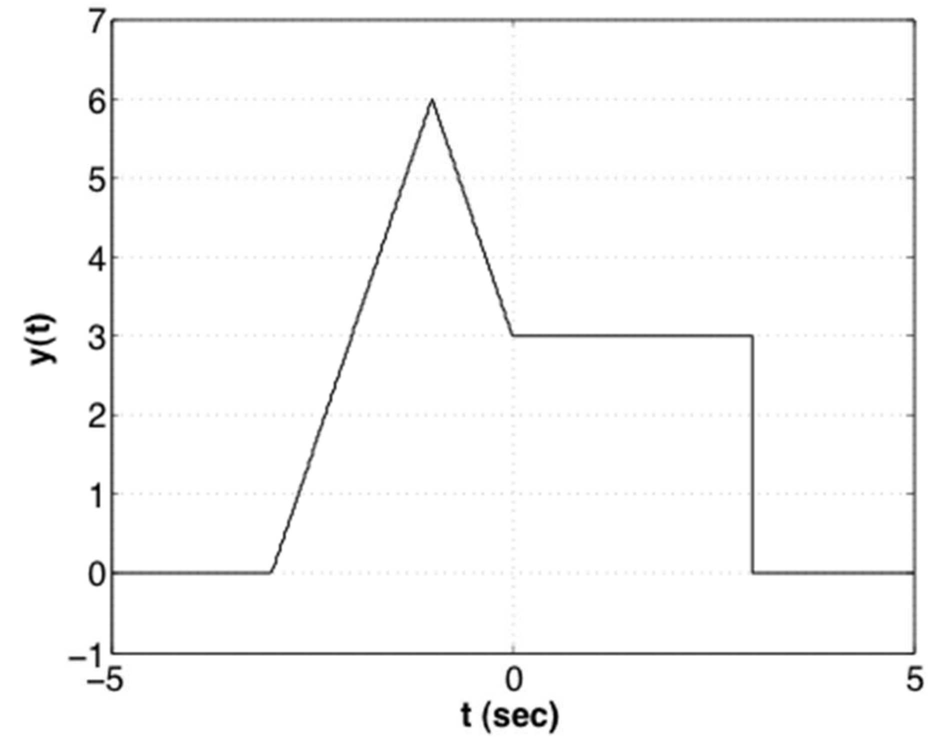
$x_2(t)$, jump discontinuities at $t = 0$, $t = 1$ and $t = 2$ so discontinuous, continuous component 0

$$\frac{dx_2(t)}{dt} = \delta(t) - 2\delta(t-1) + \delta(t-2)$$



$$y(t) = 3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3)$$

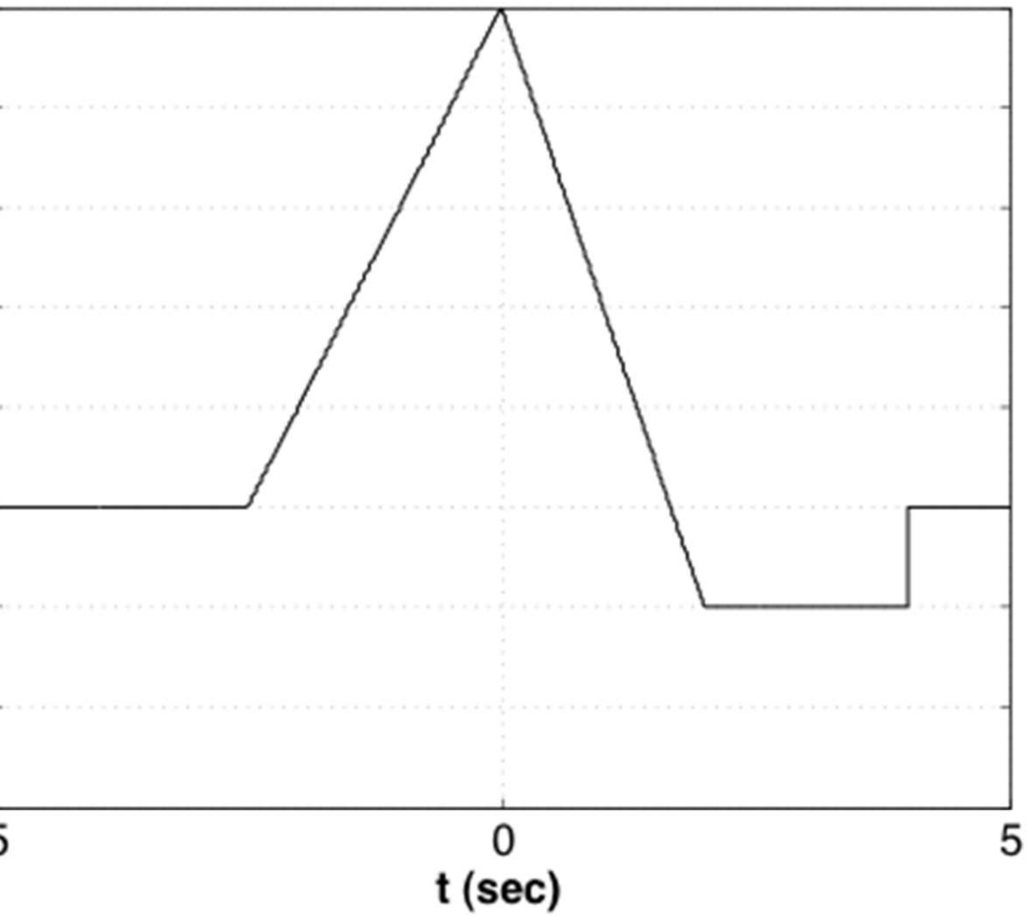
plot it and verify analytically that the obtained figure is correct.

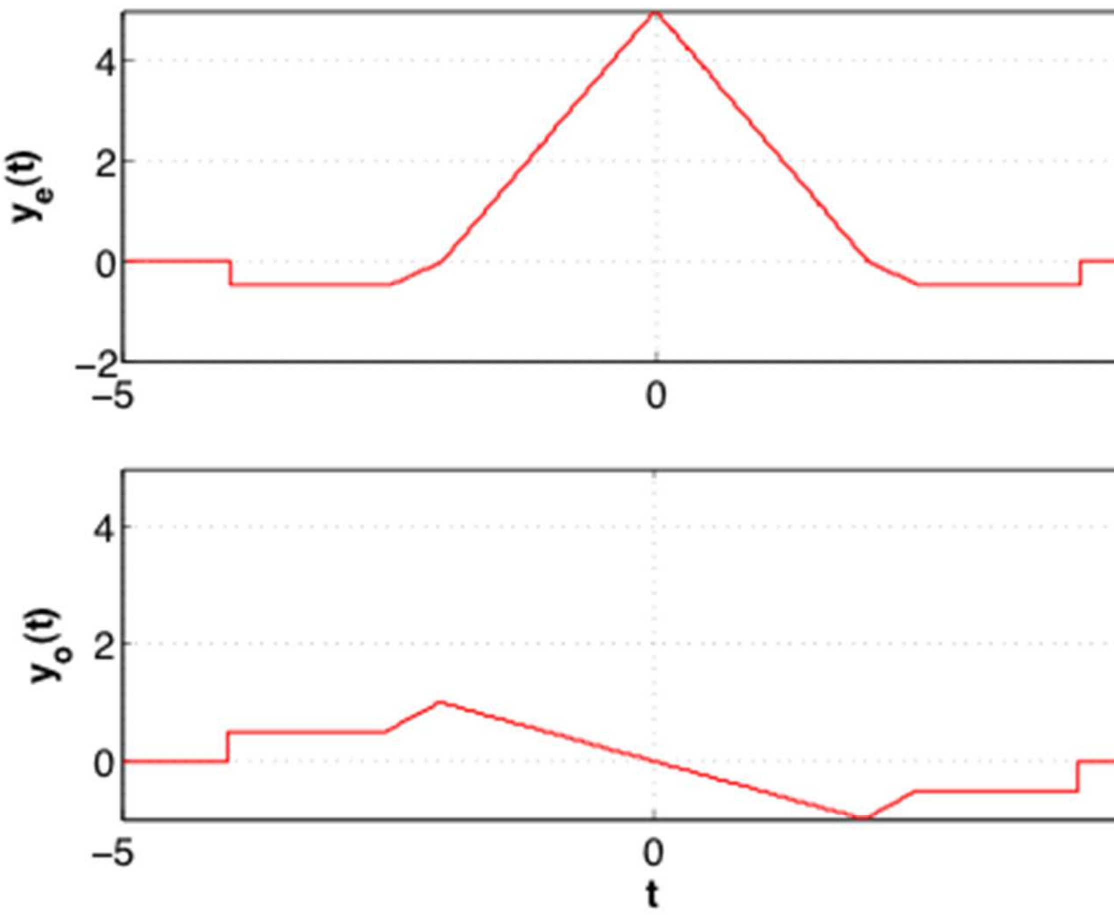


Analytically,

- $y(t) = 0$ for $t < -3$ and for $t > 3$, signal in support $-5 \leq t \leq 5$
- $-3 \leq t \leq -1$, $y(t)$ is $3r(t+3) = 3(t+3)$ which is $y(-3) = 0$, $y(-1) = 6$
- $-1 \leq t \leq 0$, $y(t)$ is $3r(t+3) - 6r(t+1) = 3(t+3) - 6(t+1) = -3t+3$, $y(-1) = 6$, $y(0) = 3$,
- $0 \leq t \leq 3$, $y(t)$ is $3r(t+3) - 6r(t+1) + 3r(t) = -3t+3+3t = 3$,
- $t > 3$, $y(t)$ is $3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3) = 3 - 3 = 0$.

components.





Signal $y(t) = 2r(t + 2.5) - 5r(t) + 3r(t - 2) + u(t - 4) - u(t - 5)$ *(left), even component* $y_e(t)$ *(right-top), odd component* $y_o(t)$ *(right-bottom).*

Example Use $r(t)$ and $u(t)$ to represent the triangular signal $\Lambda(t)$ and its derivative.

$$\Lambda(t) = \begin{cases} t & 0 \le t \le 1 \\ -t+2 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

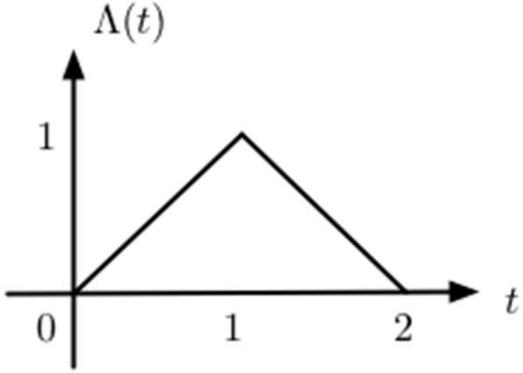
$$\Lambda(t) = r(t) - 2r(t-1) + r(t-2)$$

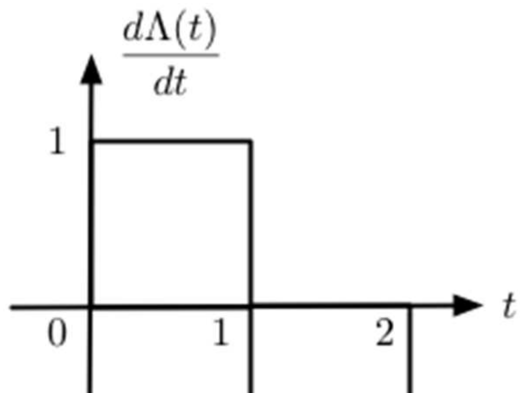
In fact,

$$\begin{aligned} \Lambda(t) &= r(t) = t \qquad \text{for} \;\; 0 \le t \le 1 \\ &= r(t) - 2r(t-1) = t - 2(t-1) = -t+2 \qquad \text{for} \;\; \le t \le 2 \\ &= r(t) - 2r(t-1) + r(t-2) = t - 2(t-1) + (t-2) = 0 \qquad t > 2 \end{aligned}$$

Derivative:

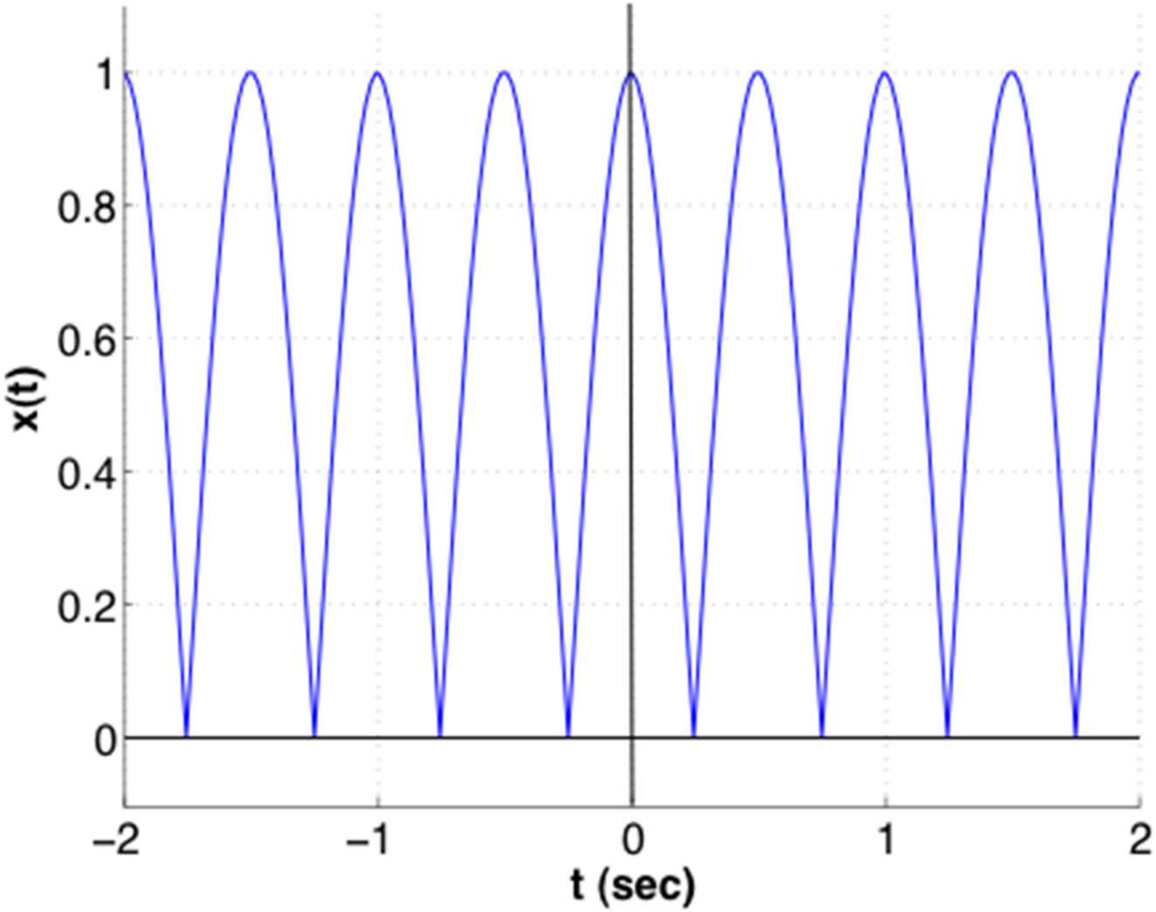
$$\frac{d\Lambda(t)}{dt} = u(t) - 2u(t-1) + u(t-2) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$





$$x(t) = |\cos(2\pi t)| \qquad -\infty < t < \infty$$

Representation for a period, and represent $x(t)$ in terms of shifted versions of it.



Solution

Period, $0 \le t \le T_0 = 0.5$:

$$p(t) = x(t)[u(t) - u(t-0.5)] = |\cos(2\pi t)|[u(t) - u(t-0.5)]$$

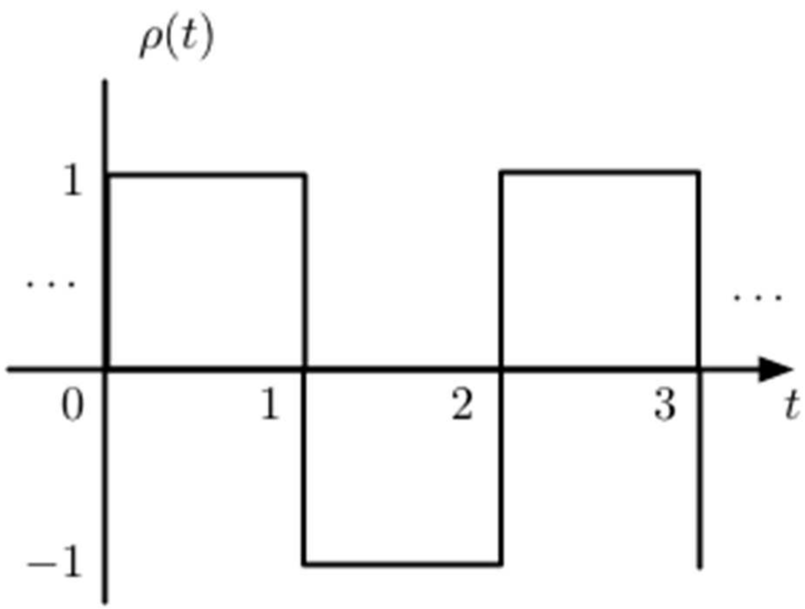
$$x(t) = \sum^{\infty} p(t - kT_0)$$

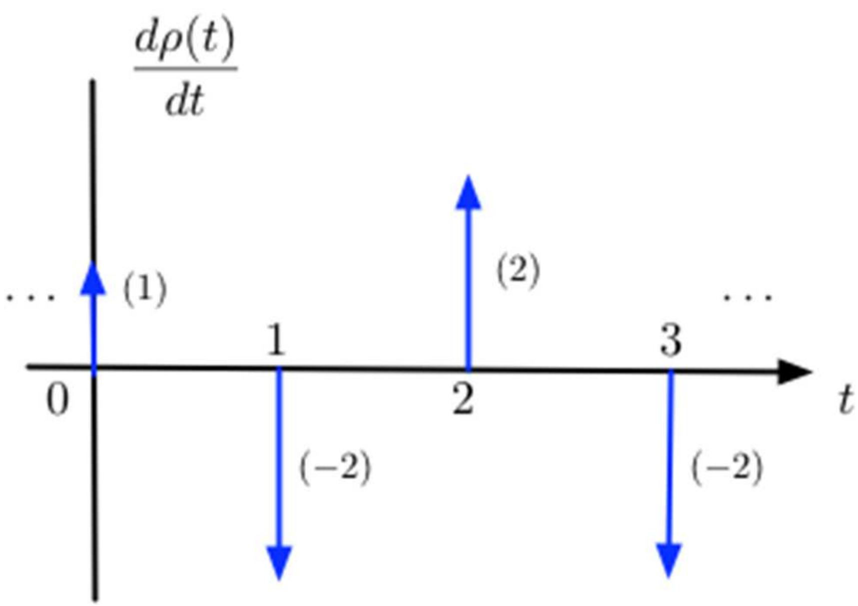
Example Generate causal train of pulses, repeating every 2 units of time using first period. Find its derivative.

$$s(t) = u(t) - 2u(t-1) + u(t-2)$$

Solution

$$\rho(t) = \sum_{k=0}^{\infty} s(t-2k)$$



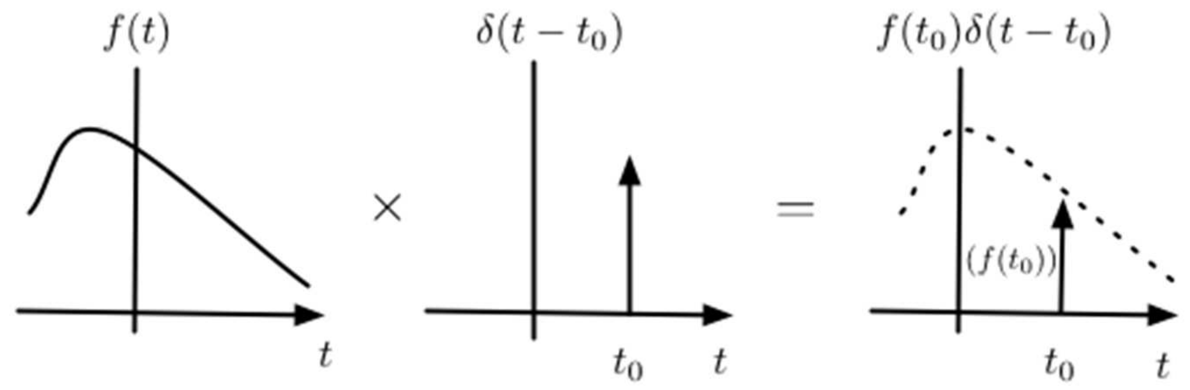


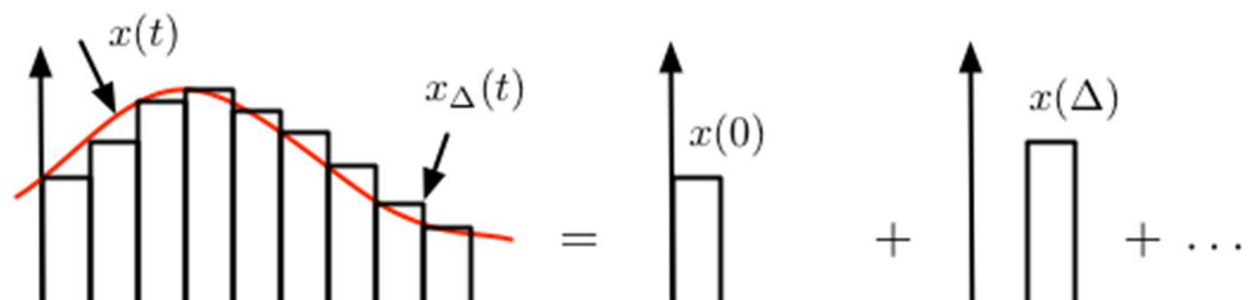
Generic Representation of Signals

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{\infty}\delta(t-\tau)dt$$
$$= f(\tau) \qquad \text{for any } \tau$$

By the sifting property of the impulse function $\delta(t)$ any signal $x(t)$ can be represented by the following **generic representation***:*

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$





Vhat have we accomplished?

- Signal classification
- Symmetry, periodicity, energy/power for continuous-time signals
- Signal representation using basic signals (unit-step, impulse, ramp, e

Where do we go from here?

- Connect signals and systems
- Develop theory that approximates behavior of most systems
- Time and frequency analysis