The Routh-Hurwitz Stability Criterion, Revisited: The Case of Multiple Poles on Imaginary Axis

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***Abstract*—This technical note presents a relation between the number of zero rows (rows with all zero elements) in the Routh array and the multiplicity of $j\omega$-axis poles. The main result of the technical note is that the existence of more than one zero row in the Routh array amounts to instability of the system irrespective of any sign change in the first column.**

***Index Terms*—Control systems, marginal stability, multiple poles on $j\omega$-axis, Routh-Hurwitz criterion.**

I. Introduction

The stability of feedback control systems is the primary concern of the control system design. As it is well known, a linear time invariant (LTI) system is stable if and only if the minimal polynomial of the dynamics matrix has no roots in the right half plane (RHP) and no multiple roots on the $j\omega$-axis. If the same conditions are verified by the characteristic polynomial of the dynamics matrix, then stability holds.

Routh-Hurwitz criterion, which was independently developed by E. J. Routh and A. Hurwitz in the late 19th century, is a simple but powerful approach to determining the number of RHP roots of a polynomial without computing those roots. Through using the coefficients of the polynomial and by a simple calculation explained in Section II of this technical note, an array is constructed. Then, the number of RHP roots is obtained based on the number of sign changes in the first column of the Routh array.

However, in determining stability, using the Routh-Hurwitz criterion, one might face certain singularities. For instance, there might be a case in which the first element of a row becomes zero. Several solutions have been offered to this case in textbooks [1]–[4] and papers [5]–[7]. In [5], the ε-method is proposed, and in [6], it is shown that this method can be applied even to such cases in which all the elements of a row go to zero as $\varepsilon \to 0$.

Another singularity that may rise in constructing the Routh array is the case where all elements of a row become zero. It is suggested that the coefficients of this row be replaced with those constructed by the derivative of the auxiliary polynomial (the polynomial corresponding to the row before the zero-row) [1]–[4], [8] so that the array will be completed. The first conclusion that might be drawn in this case is that the polynomial has symmetric roots with respect to the $j\omega$-axis [1], [7]. Once the array is completed, the location of the symmetric roots can be determined. They might be on the right and left and/or on the $j\omega$-axis.

If the symmetric roots are on the $j\omega$-axis, the best conclusion one can expect for the system is simple stability. However, **no conclusion** can be drawn regarding system stability since there might be multiple roots on the $j\omega$-axis, which is another source of instability.

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TABLE I
The Routh-Hurwitz Array

s^n	b_{n1}	b_{n2}	b_{n3}	$\cdots$
s^{n-1}	$b_{(n-1)1}$	$b_{(n-1)2}$	$b_{(n-1)3}$	$\cdots$
s^{n-2}	$b_{(n-2)1}$	$b_{(n-2)2}$	$b_{(n-2)3}$	$\cdots$
s^{n-3}	$b_{(n-3)1}$	$b_{(n-3)2}$	$b_{(n-3)3}$	$\cdots$
.	.			
.	.			
s^0	a_0			

Determining system instability in the case of multiple roots on the $j\omega$-axis is known to be the main drawback of the Routh-Hurwitz criterion [2], [4], [9]. Recent editions of some control textbooks make the reader aware of the possibility of multiple $j\omega$-axis roots (e.g., [1], [2]); however, to the best of our knowledge, no textbook and/or paper has provided a simple methodology that gives information about the number of $j\omega$-axis roots with multiplicity greater than one from the Routh array without actually solving the auxiliary polynomial. In [8], a method is provided to count the number of $j\omega$-axis roots of a complex polynomial and their multiplicity. Note that the Routh array of complex polynomials is quite different from that of real polynomials. Moreover, the method and the proof which is based on the Sturm theorem and Cauchy indices are extremely complicated as compared to the results presented in this technical note.

In this technical note, a relation is extracted between the number of zero rows in the Routh array and the multiplicity of symmetric poles, demonstrating that the stability/instability conclusion can be drawn from the Routh-Hurwitz criterion even in the cases where the existence of multiple poles on the $j\omega$-axis is the only source of instability. The rest of the technical note is organized as follows. In Section II, the details of the Routh criterion leading to zero rows are given. Then, in Section III, the main theorems and results of this technical note are presented including the proofs. Section IV includes numerical examples to support the theory. Finally, Section V gives the conclusions.

II. The Cause of Appearance of Zero Rows in Routh Array

Consider a system whose characteristic equation is given by the following monic polynomial:

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \forall i a_i \in \mathbb{R}, a_0 \neq 0. \quad (1)$$

If $a_0 = 0$, then we can simply write $\Delta(s) = s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1) = s\Delta_1(s)$ and construct the array for $\Delta_1(s)$. It is clear that if the system has more than one root at the origin, i.e., ($a_i = 0$, $i = 0, 1, ..r, 1 \leq r < n-1$), then the system is unstable and marginal stability is out of question.

The Routh array corresponding to (1) can be formed as follows [1]–[4]:

where

$$\begin{cases} \{b_{n1} = 1,\ b_{nj} = a_{n+2-2j},\ j = 2, 3, \ldots \\ b_{(n-1)j} = a_{n+1-2j},\ j = 1, 2, 3, \ldots \\ b_{ij} = -\dfrac{\begin{vmatrix} b_{(i+2)1} & b_{(i+2)(j+1)} \\ b_{(i+1)1} & b_{(i+1)(j+1)} \end{vmatrix}}{b_{(i+1)1}},\ \begin{cases} 0 \leq i \leq (n-2) \\ j = 1, 2, 3, \ldots \end{cases} \end{cases}. \quad (2)$$

The row corresponding to s^i consists of the coefficients of the following polynomial:

$$D_i(s) = b_{i1}s^i + b_{i2}s^{i-2} + b_{i3}s^{i-4} + \cdots. \quad (3)$$

Equation (3) demonstrates that a particular row includes only even or odd terms. To understand the details behind the Routh criterion, let us rewrite the characteristic (1) in the following form:

$$\Delta(s) = D_n(s) + D_{n-1}(s) \tag{4}$$

where

$$D_n(s) = s^n + a_{n-2}s^{n-2} + \cdots \tag{5}$$
$$D_{n-1}(s) = a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + \cdots . \tag{6}$$

$D_n(s)$ and $D_{n-1}(s)$ correspond to the rows s^n and s^{n-1}, repectively.

In fact, the procedure of constructing the Routh array involves successive divisions of $D_i(s)$ by $D_{i-1}(s)$ [3]:

$$\frac{D_n(s)}{D_{n-1}(s)} = \gamma_1 s + \frac{D_{n-2}(s)}{D_{n-1}(s)}. \tag{7}$$

Therefore

$$\Delta(s) = (\gamma_1 s + 1)D_{n-1}(s) + D_{n-2}(s) \tag{8}$$

where $\gamma_1 s$ and $D_{n-2}(s)$ are the quotient and remainder of the division, respectively.

The procedure is continued in the following way:

$$\frac{D_n(s)}{D_{n-1}(s)} = \gamma_1 s + \frac{1}{\frac{D_{n-1}(s)}{D_{n-2}(s)}} = \gamma_1 s + \frac{1}{\gamma_2 s + \frac{1}{\frac{D_{n-2}(s)}{D_{n-3}(s)}}} = \cdots \tag{9}$$

and

$$\Delta(s) = [(\gamma_1 s + 1)\gamma_2 s + 1] D_{n-2}(s) + (\gamma_1 s + 1) D_{n-3}(s) = \cdots . \tag{10}$$

Thus, if the polynomial $D_{i+1}(s)$ is divisible by $D_i(s)$, the remainder $D_{i-1}(s)$ becomes zero, making the row corresponding to s^{i-1} be a zero row. Therefore, we have

$$D_{i+1}(s) = \gamma_{(n-i)} s D_i(s) \tag{11}$$

and from (10), we can get

$$\Delta(s) = R(s) D_i(s) = D_i(s)(R_{n-i}(s) + R_{n-1-i}(s)). \tag{12}$$

From (11), we can conclude that the polynomial $D_{i-1}(s)$ cannot be even because of the presence of $a_0 \neq 0$ at the end of all even rows in the array. Hence, it can be inferred that in the case of $a_0 \neq 0$, zero rows can occur only in the odd rows. Since $D_i(s)$ is an even polynomial, its roots are symmetric with respect to the $j\omega$-axis. Thus, it can only contain the following types of factors:

$$\text{(I) } (s^2 + \omega^2), \text{ (II) } (s^2 - \sigma^2), \text{ (III) } (s \pm \sigma)^2 + \omega^2. \tag{13}$$

As can be seen from (12), all factors of $D_i(s)$ are also factors of $\Delta(s)$. To complete the table, one has to replace $D_{i-1}(s)$ by $D_i'(s)$, the derivative of $D_i(s)$ with respect to s.

III. Main Results

The following lemmas present the main results of the technical note, which state the relation between the number and location of zero rows and the multiplicity of $j\omega$-axis roots, helping us to examine the marginal stability of the system.

Lemma 1: An LTI system with characteristic polynomial given in (1) is unstable if there is more than one zero row in its corresponding Routh array, irrespective of any sign change in the first column.

Proof: on the event of a zero row, its coefficients would be replaced by $D_i'(s)$. Now, let $D_{i-2}(s)$ be the remainder of the division of $D_i(s)$ by $D_i'(s)$, and so on. Moreover, based on (12), it is obvious that $D_i(s)$ is the greatest common divisor (g.c.d.) of the polynomials $D_n(s)$ and $D_{n-1}(s)$. Similarly, if the next zero row corresponds to $D_{i-j-1}(s)$, we can conclude that $D_{i-j}(s)$, which includes only symmetric factors, is the g.c.d. of $D_i(s)$ and $D_i'(s)$. Hence, its symmetric roots are the repeated roots of $D_i(s)$ as well as $\Delta(s)$, because the common roots of a function and its derivative can be considered the repeated roots of that function. The next zero row represents the g.c.d. of $D_i(s)$, $D_i'(s)$, and $D_i''(s)$, and so on. Consequently, if m zero rows exist, there is at least one symmetric factor with multiplicity order of m, which can be deduced as the maximum order of multiplicity. No matter what type this symmetric factor is (see (13)), the system is unstable. In other words, type (II) or (III), due to RHP poles, are unstable themselves, and type (I) is unstable when it is repeated.

Lemma 2: An LTI system with characteristic polynomial given in (1) is marginally stable if and only if there is merely one zero row in its Routh array without any sign change in the first column.

Proof: When there is no other zero row after $D_{i-1}(s)$, we can conclude that $D_i(s)$ and $D_i'(s)$ have no common factor and all symmetric roots of $D_i(s)$ are simple. In this case, if there is no sign change in the first column, the system is marginally stable due to some simple pairs of conjugate poles on the $j\omega$-axis.

Definition.1: Let the distance d_j be the number of rows between the two consecutive zero rows, i.e., the $(j+1)^{\text{th}}$ zero row will occur d_j rows after the j^{th} zero row, and η_j be the number of sign changes between these two zero rows. For the last zero row, let d_j be the number of rows from the last zero row (inclusive) to the end of the array.

Lemma 3: Consider the characteristic polynomial given in (1), the corresponding Routh array as completed in Table I, and the definition above for d_j and η_j. If d_{j+1} is reduced with respect to d_j by 2, but η_{j+1} is equal to η_j, we can conclude that there is one pair of conjugate roots on the $j\omega$-axis with the degree of multiplicity equal to j.

Proof: Consider (1), and assume that $\Delta(s)$ has several symmetric roots with different orders of multiplicity as follows:

$$\begin{aligned}\Delta(s) &= [(s^2+\omega_1^2)^{m_1}\cdots(s^2+\omega_k^2)^{m_k}] \\ &\quad\cdot[(s^2-\sigma_1^2)^{l_1}\cdots(s^2-\sigma_p^2)^{l_p}] \\ &\quad\cdot[(s^4+2\xi_1 s^2+\gamma_1^2)^{h_1}\cdots(s^4+2\xi_q s^2+\gamma_q^2)^{h_q}]R(s) \\ &= D_i(s)R(s) = D_i(s)(R_{n-i}(s)+R_{n-1-i}(s))\end{aligned} \tag{14}$$

where all symmetric poles are shown separately with their multiplicity; hence, the polynomial $R(s)$ does not have any symmetric root.

In (14)

$$\omega_j \neq 0 \text{ and } 1 \le m_j \le m_{j-1}, \quad j = 2,3,\ldots,k$$
$$\sigma_j \neq 0 \text{ and } 1 \le l_j \le l_{j-1}, \quad j = 2,3,\ldots,p$$
$$\gamma_j \neq 0 \text{ and } 1 \le h_j \le h_{j-1}, \quad j = 2,3,\ldots,q.$$

Consider a completed Routh array for $\Delta(s)$ given in (1). Let, $\Delta_i(s)$ be the polynomial constructed from the two adjacent rows of the array, e.g., $\Delta_i(s) = D_i(s) + D_{i-1}(s)$. It is known that the Routh array corresponding to $\Delta_i(s)$ is exactly the same as that of the original array ($\Delta(s)$), from the row corresponding to $D_i(s)$ onward [1], [10], [11]. This is also true for the case of a zero row, i.e., when $D_{i-1}(s)$ is zero.

Hence, the sign variations in the first column prior to the first zero row is due to the existence of RHP roots, the LHP symmetries of which are not the roots of $\Delta(s)$. In other words, these RHP roots belong to $R(s)$ in (12). Now, let us define $\Delta_i(s)$ as follows:

$$\begin{aligned}\Delta_i(s) &= D_i(s) + D_i'(s)\\ &= \left[\left(s^2+\omega_1^2\right)^{m_1-1}\ldots\left(s^2+\omega_k^2\right)^{m_k-1}\right]\\ &\quad\cdot\left[\left(s^2-\sigma_1^2\right)^{l_1-1}\ldots\left(s^2-\sigma_p^2\right)^{l_p-1}\right]\\ &\quad\cdot\left[\left(s^4+2\xi_1 s^2+\gamma_1^2\right)^{h_1-1}\ldots\left(s^4+2\xi_q s^2+\gamma_q^2\right)^{h_q-1}\right]\\ &\quad\cdot\left(R_{d_1}(s)+R_{d_1-1}(s)\right)\\ &= D_{i-d_1}(s)\left(R_{d_1}(s)+R_{d_1-1}(s)\right) \qquad (15)\\ d_1 &= 2k+2p+4q \qquad (16)\end{aligned}$$

where k is the number of the pairs of conjugate poles on $j\omega$-axis, and p and q are the number of type (II) and type (III) factors, respectively.

By comparing (15) with (12) and (14), we can conclude that the second zero row will occur at the row corresponding to $D_{i-d_1-1}(s)$. The number of sign changes between these two zero rows is exactly the number of RHP roots of $R_{d_1}(s)$. This is due to the fact that the part of the array from the row of $D_{i-d_1}(s)$ onward is the same as the Routh array corresponding to $D_{i-d_1}(s)$ (see the discussion above), and that $D_i(s) = R_{d_1}(s)D_{i-d_1}(s)$. Therefore

$$\eta_1 = p + 2q. \qquad (17)$$

From (16) and (17), we have

$$k = \frac{d_1}{2} - \eta_1. \qquad (18)$$

d_j and η_j remain constant unless the differentiating operation eliminates one of the factors which has a lower degree of multiplicity than the others. Elimination of a factor of type (I) only reduces d_j by two units but does not change η_j. Elimination of a factor of type (II) reduces d_j by two and η_j by one unit, and finally, elimination of a factor of type (III) reduces d_j by four and η_j by two units. Sometimes, it is difficult to distinguish between types (II) and (III), because elimination of two factors of type (II) has a result identical to the elimination of a factor of type (III). Fortunately, elimination of $j\omega$-axis roots has different results and is distinguishable from other types. In other words, reduction of distance without reduction of sign variations after the $(j+1)^{\text{th}}$ zero row means there is a pair of conjugate roots on the jω-axis with repeated order of j, which completes the proof.

IV. Examples

Although these examples do not have practical aspects, they are provided to exemplify the theorems and the procedure of counting the number and repeated order of the pairs of conjugate poles on the $j\omega$-axis.

In the examples given below, the highlighted rows are zero rows whose coefficients are replaced by the coefficients of the derivatives of the corresponding auxiliary polynomials.

1) Example 1:

$$\begin{aligned}\Delta(\boldsymbol{s}) &= s^{16}+2s^{15}+14s^{14}+26s^{13}+83s^{12}+140s^{11}\\ &\quad+272s^{10}+404s^9+539s^8+674s^7+662s^6+650s^5\\ &\quad+493s^4+336s^3+204s^2+72s+36.\end{aligned}$$

TABLE II
The Routh Array for Example 1

16	1	14	83	272	539	662	493	204	36
15	2	26	140	404	674	650	336	72	0
14	1	13	70	202	337	325	168	36	0
13	14	156	700	1616	2022	1300	336	0	0
12	1.8571	20	86.5714	192.5714	232.1429	144	36	0	0
11	5.2308	47.3846	164.3077	272.0000	214.4615	64.6154	0	0	0
10	3.1765	28.2353	96.0000	156.0000	121.0588	36	0	0	0
9	0.8889	6.2222	15.1111	15.1111	5.3333	0	0	0	0
8	6.0000	42.0000	102.0000	102.0000	36.0000	0	0	0	0
7	48.0000	252.0000	408.0000	204.0000	0	0	0	0	0
6	10.5000	51.0000	76.5000	36.0000	0	0	0	0	0
5	18.8571	58.2857	39.4286	0	0	0	0	0	0
4	18.5455	54.5455	36.0000	0	0	0	0	0	0
3	2.8235	2.8235	0	0	0	0	0	0	0
2	36.0000	36.0000	0	0	0	0	0	0	0
1	72.0000	0	0	0	0	0	0	0	0
0	36.0000	0	0	0	0	0	0	0	0

TABLE III
The Routh Array for Example 2

16	1	-3	5	-9	6	0	-4	12	-8
15	-1	1	-3	3	0	0	4	-4	0
14	-2	2	-6	6	0	0	8	-8	0
13	-28	24	-60	48	0	0	16	0	0
12	0.2857	-1.7143	2.5714	0	0	6.8571	-8	0	0
11	-144	192.0000	48	0	672	-768	0	0	0
10	-1.3333	2.6667	0	1.3333	5.3333	-8	0	0	0
9	-96.0000	48.0000	-144.0000	96.0000	96.0000	0	0	0	0
8	2.0000	2.0000	0	4.0000	-8	0	0	0	0
7	144.0000	-144.0000	288.0000	-288.0000	0	0	0	0	0
6	4.0000	-4.0000	8.0000	-8	0	0	0	0	0
5	24	-16	16	0	0	0	0	0	0
4	-1.3333	5.3333	-8	0	0	0	0	0	0
3	80.0000	-128.0000	0	0	0	0	0	0	0
2	3.2000	-8	0	0	0	0	0	0	0
1	72.0000	0	0	0	0	0	0	0	0
0	-8	0	0	0	0	0	0	0	0

Based on Lemma 1, the system corresponding to this array is unstable, because it has three zero rows $(m = 3)$, despite having no sign variation in the first column $(\eta = 0)$.

The distance between the first two zero rows is 6 rows $(d_1 = 6)$), and hence $k = (d_1/2) - \eta = 3$, i.e., the characteristic equation has **three** pairs of $j\omega$-axis conjugate poles. According to (18), $m_1 = m = 3$, which means that at least one of these three pairs has a multiplicity of order 3.

The distance between the second and third zero row is 6, which is the same as the distance between the first and second zero row; however, $d_3 = 2$, which is reduced by 4. Therefore, based on Lemma 3, there is two pairs of conjugate poles on the $j\omega$-axis with repeated order of 2 (one unit less than the index of d_3), i.e., $m_2 = m_3 = 2$.

Now, by looking at the roots of $\Delta_1(s)$, which are $\{-1, \pm j\sqrt{3}$ and $\pm j\sqrt{2}\}$ with a multiplicity of order 2, and $\{\pm j\}$ with a multiplicity of order 3, we can infer that the results of Lemma 1 and Lemma 2 are verified.

2) Example 2:

$$\begin{aligned}\Delta_2(\boldsymbol{s}) &= s^{16}-s^{15}-3s^{14}+s^{13}+5s^{12}-3s^{11}\\ &\quad-9s^{10}+3s^9+6s^8-4s^4+4s^3+12s^2-4s-8.\end{aligned}$$

Before the first zero row, there is one sign change and hence one RHP pole. $d_1 = 8$ and $\eta_1 = 3$, so by using (17), we can conclude that $k = 1$, where k is the number of $j\omega$-axis poles. $d_2 = 6 = d_1 - 2$, but $\eta_2 = 3 = \eta_1$; therefore, the power of the type (I) factor is 1. Thus, we only have one simple pair of conjugate poles on the $j\omega$-axis. The other 12 poles are symmetric with respect to the $j\omega$-axis on the right and left half planes. By obtaining roots of $\Delta_2(s)$, and rewriting it in the form of its factors, we have $\Delta_2(s) = \left(s^4+2\right)^2\left(s^2-1\right)^2(s^2+1)(s-2)(s+1)$, which verifies our results.

V. Conclusion

Stability analysis of LTI systems is the most important problem in linear control systems for which some methods are introduced in the textbooks. Routh-Hurwitz criterion is one of these methods, which can simply give stability/instability conclusion of a system. Although, so far, it seemed that Routh-Hurwitz criterion can only examine RHP poles and cannot give any information about multiple poles on the $j\omega$-axis , which is another source of instability, it is shown in this technical note that this criterion not only can distinguish this source of instability, but also gives adequate information about symmetric poles (including $j\omega$-axis poles).

The number of $j\omega$-axis poles and their multiplicity can be calculated in every case, based on the distances and the number of sign changes between the zero rows.

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Correction to "Packetized Predictive Control of Stochastic Systems Over Bit-Rate Limited Channels With Packet Loss"

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In [1], we showed that a particular class of networked control system (NCS) with quantization, i.i.d. dropouts and disturbances can be described as a Markov jump linear system of the form

$$\theta_{k+1} = \bar{A}(d_k)\theta_k + \bar{B}(d_k)\nu_k \tag{1}$$

where

$$\theta_k \triangleq \begin{bmatrix} x_k \\ b_{k-1} \end{bmatrix} \in \mathbb{R}^{n+N}, \quad \nu_k \triangleq \begin{bmatrix} w_k \\ n_k \end{bmatrix} \in \mathbb{R}^{m+N}$$

and $\{d_k\}_{k\in\mathbb{N}_0}$ is a Bernoulli dropout process, with

$$\text{Prob}(d_k = 1) = p \in (0, 1).$$

Throughout [1] we showed that properties of the NCS can be conveniently stated in terms of the expected system matrices

$$\mathcal{A}(p) = \mathbb{E}\{\bar{A}(d_k)\}$$
$$\mathcal{B}(p) = \mathbb{E}\{\bar{B}(d_k)\} = [\,\mathcal{B}_w \quad \mathcal{B}_n(p)\,]$$

and the matrix $\widetilde{\mathcal{A}} = \bar{A}(1) - \bar{A}(0)$. Unfortunately, Theorem 4 in [1, Section V-A] is incorrect. For white disturbances $\{w_k\}_{k\in\mathbb{N}_0}$, the statement should be as given below. Non-white $\{w_k\}_{k\in\mathbb{N}_0}$ can be accommodated by using standard state augmentation techniques; see, e.g., [2].

Theorem 4: Suppose that (1) is MSS and AWSS and that $\{w_k\}_{k\in\mathbb{N}_0}$ is white with $\sigma_w^2 = \text{tr}R_w(0)$. Define

$$\mathcal{F}(z) \triangleq \left(zI - \mathcal{A}(p)\right)^{-1}$$
$$\mathcal{C}(p) \triangleq \left(\frac{\sigma_w^2}{m}\right)\mathcal{B}_w\mathcal{B}_w^T + \left(\frac{\sigma_n^2}{N}\right)(1-p)\mathcal{E} \in \mathbb{R}^{(n+N)\times(n+N)} \tag{2}$$

where (see [1, Sec.2] for definitions)

$$\mathcal{E} \triangleq \frac{\mathcal{B}_n(p)\mathcal{B}_n(p)^T}{(1-p)^2} = \begin{bmatrix} B_1 e_1^T(\Psi^T\Psi)^{-1}e_1 B_1^T & B_1 e_1^T(\Psi^T\Psi)^{-1} \\ (\Psi^T\Psi)^{-1}e_1 B_1^T & (\Psi^T\Psi)^{-1} \end{bmatrix}. \tag{3}$$

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