A logical approach to uniformity in Diophantine geometry

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What is a finiteness theorem?

By a finiteness theorem of Diophantine geometry I generally mean a theorem to the effect that some given system of algebraic equations has only finitely many solutions or at least that the solution set takes a particularly simple form when we insist that the solutions come from some specified arithmetically meaningful set.

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This theorem scheme requires some explanation as to the meanings of most of the highlighted terms.

Example: Mordell's conjecture

Theorem (Faltings)

Let $F(X, Y) \in \mathbb{Q}[X, Y]$ be an irreducible polynomial in two variables with rational coëfficients of total degree at least four. Then there are at most finitely many pairs of rational number $(a, b) \in \mathbb{Q}^2$ for which $F(a, b) = 0$.

Example: Unit equation

Theorem (Lang)

Let K be a number field and $U_K := \mathscr{O}_K^\times$ the group of units in the ring of integers of K (the set of elements $a \in K$ for which both a and a^{-1} are zeroes of a monic polynomial with integer coëfficients). If α and β are two elements of K, then there are only finitely many solutions to $\alpha x + \beta y = 1$ with $x, y \in U_K$.

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In this case "finite" really means finite. There are no exceptional systems of equations.

Example: Unit equation, generalized

As before, take K to be a number field and U_K to be the group of units in its ring of integers but consider an arbitrary system of equations in several variables.

$$f_1(x_1, \ldots, x_n) = \cdots = f_\ell(x_1, \ldots, x_n) = 0 \quad (\dagger)$$

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Theorem (Lang)

The solutions to $(\dagger)$ with $(x_1, \ldots, x_n) \in U_K^n$ lie in a finite union of solution sets to special systems of equations.

Example: André-Oort

Conjecture

If $F(X, Y) \in \mathbb{C}[X, Y]$ *is an irreducible polynomial over the complex numbers and* $F(a, b) = 0$ *for infinitely many pairs* (a, b) *where each coördinate is the* j*-invariant of an elliptic curve with complex multiplication, then* F *is a* modular *polynomial.*

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The above conjecture is equivalent to a finiteness theorem in our sense of the form *for any nonzero polynomial G in two variables over the complex numbers, there are finitely many modular polynomials $F_1, \ldots, F_\ell$ dividing G so that the set of solutions to $G(x, y) = 0$ with (x, y) a pair of CM j-invariants differs from that to $F_1 \cdots F_\ell(x, y) = 0$ by a finite set.*

What is uniform finiteness?

In the case of a finiteness theorem for which we expect an outright finite number of solutions, for the uniform version we would ask that the finite number of solutions in question to be bounded by a function of the degrees of the polynomials in the system of equations.

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When we allow special systems of equations in the conclusion of the theorem, then in the uniform version we ask that the degrees of the relevant special equations also be bounded as a function of the degrees of the initial system.

Nonstandard methods

Question

Can one derive a uniform version of Faltings' Theorem (Mordell's Conjecture) by considering nonstandard models of $\mathrm{Th}(\mathbb{Q}, +, \times)$*?*

- If the uniform version were false, then by the compactness theorem we could find an elementary extension ${}^*\mathbb{Q} \succeq \mathbb{Q}$ and an irreducible polynomial $F(X, Y) \in {}^*\mathbb{Q}[X, Y]$ of degree at least four having infinitely many zeroes in ${}^*\mathbb{Q}$.
- As ${}^*\mathbb{Q} \equiv \mathbb{Q}$, Faltings' Theorem interpreted in ${}^*\mathbb{Q}$ says that there are boundedly many solutions to $F(x, y) = 0$.
- At this point, one might hope to use something about nonstandard models of arithmetic.
- It bears noting, that A. Robinson, in his final paper published posthumously, considered a similar strategy towards a proof of Mordell's Conjecture.

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Why does the compactness argument fail?

- The main problem is that *finite* is not a first-order property. That the number of solutions is pseudofinite (or internally finite) may be meaningful, but it is not obvious what this meaning is.
- On the face of it, the finiteness theorems in question only give information about the standard model, but if the compactness argument is to help we need to know something about the nonstandard models as well.

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Stability

Definition

Let $\mathfrak{M}$ be an $\mathscr{L}$-structure for some first-order language $\mathscr{L}$, $A \subseteq M = |\mathfrak{M}|$ some subset of the universe of $\mathfrak{M}$, and $n \in \mathbb{Z}_+$ a positive integer. An *n-type over A* is a maximal consistent theory in $\mathscr{L}_A(x_1, \ldots, x_n)$ extending the $\mathscr{L}_A$-theory of $\mathfrak{M}$. The set of n-types over A is denoted $S_n(A)$.

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An $\mathscr{L}$-theory T is *stable* if for arbitrarily large cardinals λ if $\mathfrak{M} \models T$ has a universe M of cardinality λ, then $|S_1(M)| = \lambda$.

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It follows from Morley's work on Łoś's Conjecture that every $\aleph_1$-categorical theory is stable. In particular, algebraically closed fields are stable.

Definability of types

Definition

Let $\mathscr{L}$ be a first-order language, $\mathfrak{M}$ an $\mathscr{L}$-structure, $A \subseteq M$ a subset of the universe of $\mathfrak{M}$, and $p \in S(A)$ a type over A. We say that p is definable over A if for each formula $\varphi(\mathbf{x};\mathbf{y})$ there are another formula $\psi(\mathbf{y};\mathbf{z})$ and parameters $\mathbf{b}$ from A so that for any tuple $\mathbf{a}$ from A one has $\varphi(\mathbf{x};\mathbf{a}) \in p$ if and only if $\mathfrak{M} \models \psi(\mathbf{a};\mathbf{b})$.

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Theorem (Shelah)

A theory is stable if and only if every finitary type over every subset of every model of the theory is definable.

Stable embedability

Proposition

If T is a stable $\mathscr{L}$-theory, $\mathfrak{M} \models T$, $A \subseteq M$ is a subset of the universe of $\mathfrak{M}$, and $X \subseteq M$ is a an $\mathscr{L}_M$-definable set, then there is an $\mathscr{L}_A$-definable set Y for which $X \cap A = Y \cap A$.

- Write $X = \{x \in M \mid \mathfrak{M} \models \varphi(\mathbf{m}, x)\}$ for an appropriate formula φ and tuple of parameters $\mathbf{m}$.
- As $p := \text{tp}(\mathbf{m}/A)$ is definable, there is a formula $\psi(x) \in \mathscr{L}_A$ for which $\varphi(\mathbf{y}, a) \in p$ if and only if $\mathfrak{M} \models \psi(a)$ for $a \in A$.
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$\square$

Uniform stable embeddability

Theorem (Shelah)

If T is a stable theory, $\mathfrak{M} \models T$, $A \subseteq M$ is a subset of the universe of $\mathfrak{M}$, and $\varphi(x, \mathbf{y})$ is any formula; then there is a formula $\psi(x, \mathbf{z})$ so that for any parameter $\mathbf{m}$ from M there is a parameter $\mathbf{b}$ from A for which $\varphi(x; \mathbf{m})$ and $\psi(x; \mathbf{b})$ define the same subsets of A.

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Proof.

If the theorem were to fail, then by applying the compactness theorem to the structure $\mathfrak{M}$ augmented by a predicate for A, we obtain a new model ${}^*\mathfrak{M}$ with a subset ${}^*\mathfrak{M}$ and a parameter $\mathbf{m}$ from ${}^*\mathfrak{M}$ for which $\{x \in A \mid {}^*\mathfrak{M} \models \varphi(x, \mathbf{m})\}$ is not the trace of an A-definable set. □

A little algebraic geometry

Definition

A complex algebraic variety is a set of the form $V = \{\mathbf{x} \in \mathbb{C}^n \mid \bigwedge f_i(\mathbf{x}) = 0\}$ where $f_1, \ldots, f_\ell$ is a sequence of polynomials in n variables with complex coëfficients.

- The algebraic varieties form the closed sets of the Zariski topology.
- A constructible set is a finite Boolean combination of varieties. By quantifier elimination for $(\mathbb{C}, +, \times)$, the constructible sets are exactly the definable sets.

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Special varieties

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Let $\Gamma \subseteq \mathbb{C}^m$ be a set. We say that the algebraic variety $X \subseteq \mathbb{C}^{mn}$ is special if $\Gamma^n \cap X$ is dense in X with respect to the Zariski topology.

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Example

For example, if Γ is the set of roots of unity, then the special varieties are all finite unions of multiplicative translates of subgroups of $(\mathbb{C}^\times)^n$.

Automatic uniformity

Theorem

Let $\Gamma \subseteq \mathbb{C}^m$ be any set. If the class of special varieties for Γ is closed under finite intersections, then Γ satisfies uniform finiteness in the sense that there is a function $f : \mathbb{N} \to \mathbb{N}$ so that if $X \subseteq \mathbb{C}^m$ is a variety defined by equations of degree at most d, then the closure of $X \cap \Gamma$ is defined by equations of degree at most $f(d)$. In particular, if $X \cap \Gamma$ is finite, then it consists of at most $f(d)$ points.

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Proof.

Uniform stable embeddness. □

The hypotheses can be relaxed slightly to asserting only that if X and Y are special and Z is a component of $X \cap Y$ which meets Γ^n, then Z is special. In this form, automatic uniformity applies to all known examples of finiteness theorems in Diophantine geometry.

A few details in a special case

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Fix

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a polynomial in $2 + (d+1)^2$ variables.

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We wish to uniformly describe the sets

$$\{(\zeta, \xi) \in \Gamma^2 \mid f(\zeta, \xi; \mathbf{b}) = 0\}$$

as the parameter $\mathbf{b}$ varies.

A few more details

By uniform definability of types, there is a formula $\psi(x, y; \mathbf{z})$ so that for any $\mathbf{b}$ there is some tuple $\mathbf{a}$ from Γ for which for any pair $(\zeta, \xi) \in \Gamma^2$

$$\mathbb{C} \models \psi(\zeta, \xi; \mathbf{a}) \iff f(\zeta, \xi; \mathbf{b}) = 0$$

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Using quantifier elimination and some easy point-set topology, one sees that ψ may be assumed to define a special variety, namely in this case, a finite union of cosets of multiplicative groups.
The solution set to $\psi(x, y; \mathbf{a})$*to* is the projection onto the first two coördinates of the intersection of the formula ψ with the coset $(\mathbb{C}^{\times})^2 \times \{\mathbf{a}\}$. As such it is easy to read off bounds on the shape of the intersection in terms of the presentation of ψ.

Some final remarks

- In practice, often, but not always, the proofs of the finiteness theorems yield effective uniformities.
- If one can formalize the problems in an appropriate first-order theory, then stronger automatic uniformities may follow.
- D. Roessler has applied this automatic uniformity theorem to give easy proofs of finiteness theorems for abelian varieties.

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