Propositional Games with Explicit Strategies

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13th Workshop on Logic, Language, Information, and Computation

Overview

1. LP: Artemov's Logic of Proofs

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2. Two games: Nim and Verification

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3. About strategies
   - LP: A logic of explicit strategies
   - Application: LP Strategies for Nim

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Basic ideas

Extend propositional logic with formula-labeling terms.

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Internalization.
Each theorem F has a term t such that $t : F$ is a theorem.

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- Variables: $x_1, x_2, x_3, \ldots$
- Constants: $c_1, c_2, c_3, \ldots$
- *Terms* built up from constants and variables using functions.
- *Formulas* are those of CL in addition to $t\!:\!F$.

LP: Artemov's Logic of Proofs

Axioms and rules

- *Classical propositional logic*, CL
  - C. Finite collection of axiom schemas
  - RC. Modus ponens: infer B from $A \supset B$ and A
- *Evidence management*
  - LP1. $u\!:\!(A \supset B) \supset (v\!:\!A \supset (u \cdot v)\!:\!B)$
  - LP2. $u\!:\!A \supset\, !u\!:\!(u\!:\!A)$
  - LP3. $u\!:\!A \vee v\!:\!A \supset (u + v)\!:\!A$
  - LP4. $u\!:\!A \supset A$
  - RLP. Constant necessitation: infer $c\!:\!A$ from constant c and axiom A

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  LP4. $u\!:\!A \supset A$

  RLP. Constant necessitation: infer $c\!:\!A$ from constant c and axiom A

Internalization Theorem.
If F is an LP theorem, there is a variable-free term t such that $t\!:\!F$ is also an LP theorem.

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A basic version of Nim

- Given three piles of sticks: (a, b, c).

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  - Choose one pile.
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- Winner: person to pick up the last stick (so that none remain in any pile).

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(the Ehrenfeucht-Fraïssé-Hintikka subformula-choosing game)

- A is a (propositional) formula written using $\neg$ and $\vee$.

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  - Case: A is $B \vee C$. Player-to-move either chooses B or else chooses C; game continues on chosen subformula with same player-to-move.

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  - Case A is $\neg B$.
    Player-to-move changes to other player; game continues on B with this new player to move.

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Truth and validity

Definition

A is true in M: True has a winning strategy in Verification on A with model M.

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Theorem

Tarskian validity agrees with Verification validity.

Embedding Nim into Verification

Associate a propositional formula to each Nim instance.

(Idea: Copy the game tree of the Nim instance.)

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  - $(a, b, c)^T$ is the Nim game (a, b, c) with the 1st player to move.
  - $(a, b, c)^F$ is the Nim game (a, b, c) with the 2nd player to move.

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  - Example: $(4, 5, 1) \xrightarrow{1} (2, 5, 1)$

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- For a, b, c not all zero,

$$(a, b, c)^T := \bigvee_{(a,b,c) \xrightarrow{1} (a',b',c')} \neg (a', b', c')^F \ ,$$

and similarly for $(a, b, c)^F$, though with T superscripts on RHS.

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True has a winning strategy in Verification on $(a, b, c)^T$ iff 1st player has a winning strategy in Nim on (a, b, c).

Embedding Nim into Verification
An example

Example. Nim game $(1, 1, 1)$.

Embedding Nim into Verification

An example

Nim	**Verification**
1st on $(1,1,1)$.	True on $(1,1,1)^T$.

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1st on $(1,1,1)$. [Pick $(0,1,1)$, $(1,0,1)$, or $(1,1,0)$.]	True on $(1,1,1)^T$. $[\neg(0,1,1)^F \vee \neg(1,0,1)^F \vee \neg(1,1,0)^F]$

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(1st waits.)	True on $\neg(0,1,1)^F$.

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(1st waits.)	True on $\neg(0,1,1)^F$.
2nd on $(0,1,1)$.	False on $(0,1,1)^F$.

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About strategies
What are they?

So Nim may be considered as a special case of Verification.

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Strategies.
A *strategy* in the Verification game on A is a function on the parse tree of A taking each non-leaf to a child.

About strategies
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So Nim may be considered as a special case of Verification.

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Strategies.
A *strategy* in the Verification game on A is a function on the parse tree of A taking each non-leaf to a child.

Winning strategies.
A *winning strategy* is a strategy that guarantees a player a win no matter the moves of the other player.

About strategies
Some slogans

- *Avoid the opponent's winning positions.*

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  In $(1, 2, 0)$, 1st player ought to avoid $(1, 0, 0)$.

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> **Surrender.** A game extension.
> On a turn, players may:
>
> - Surrender or
> - Make a legal move.

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Some slogans

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Strategies (again).
A *strategy* in Verification on A is a function on the parse tree of A taking each non-leaf either to a child or to a unique surrender value.

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Some slogans

- *Avoid the opponent's winning positions.*
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Note: Verification on $\neg A$.
Player-to-play may

- Continue to play, waiting for the other player's response on A, or
- Surrender.

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About strategies
Some slogans

- *Avoid the opponent's winning positions.*
- *Surrender if all is lost, otherwise fight.*
- *Choose the best plan.*
  In $(1, 2, 0)$, the 1st player ought to choose his first move wisely.

About strategies

Making semi-formal sense of the slogans

Let $t : A$ mean "t is a strategy on A."

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- *Avoid the opponent's winning positions.*
  - Suppose $u : (\neg A \vee B)$, $v : A$, and both u and v are winning strategies. . .
  - Then $u : B$ and u is winning on B.

Strategy $u \cdot v$: "if $u : (\neg A \vee B)$ and $v : A$, then follow u on B."

About strategies

Making semi-formal sense of the slogans

- *Avoid the opponent's winning positions.*

  Strategy $u \cdot v$**:** "if $u:(\neg A \vee B)$ and $v:A$, then follow u on B."

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About strategies

Making semi-formal sense of the slogans

- *Avoid the opponent's winning positions.*

  Strategy $u \cdot v$**:** "if $u:(\neg A \vee B)$ and $v:A$, then follow u on B."

- *Surrender if all is lost, otherwise fight.*
  - If $u:A$ and u is a winning strategy, then one ought to follow u on A. Otherwise, if there's no other choice, one ought to give up.

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Preliminary notes

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- Subformula A.

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- Subformula B.

  Player-to-move retains turn on B.

A logic of explicit strategies
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LP4. $u : A \supset A$

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RLP. Constant necessitation: infer $c : A$ from constant c and axiom A

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Theorem (Soundness)

True has a winning strategy on each LP *theorem.*

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Theorem (Completeness)

True has a winning strategy only on LP *theorems.*

Application: LP Strategies for Nim
Internalization example

1st has a winning strategy in Nim on $(1,2)$ $\big[= (1,2,0)\big]$.

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$$(1, 2)^T = \neg(0, 2)^F \vee \neg(1, 0)^F \vee \neg(1, 1)^F$$

1. $\neg\neg\bot \supset \bot$
2. $(\neg\neg\bot \supset \bot) \supset ((\neg\neg\bot \supset \bot) \supset (\neg\neg\bot \vee \neg\neg\bot \supset \bot))$
3. $\neg\neg\bot \vee \neg\neg\bot \supset \bot$
4. $(\neg\neg\bot \vee \neg\neg\bot \supset \bot) \supset \neg(\neg\neg\bot \vee \neg\neg\bot)$
5. $\neg(\neg\neg\bot \vee \neg\neg\bot)$
6. $\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0, 2)^F \vee \neg(1, 0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))$
7. $(\neg(0, 2)^F \vee \neg(1, 0)^F) \vee \neg \underbrace{(\neg\neg\bot \vee \neg\neg\bot)}_{(1,1)^F}$

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1′. $c_1 : (\neg\neg\bot \supset \bot)$
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3. $\neg\neg\bot \vee \neg\neg\bot \supset \bot$
4. $(\neg\neg\bot \vee \neg\neg\bot \supset \bot) \supset \neg(\neg\neg\bot \vee \neg\neg\bot)$
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5'. & (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)) : \neg(\neg\neg\bot \vee \neg\neg\bot) \\
6. & \neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot)) \\
7. & (\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg\underbrace{(\neg\neg\bot \vee \neg\neg\bot)}_{(1,1)^F}
\end{array}$$

Application: LP Strategies for Nim

Internalization example

1st has a winning strategy in Nim on $(1,2)$ $\big[= (1,2,0)\big]$.

$$(1,2)^T = \neg(0,2)^F \vee \neg(1,0)^F \vee \neg(1,1)^F$$

$$\begin{array}{ll}
1'. & c_1 : (\neg\neg\bot \supset \bot) \\
2'. & c_2 : \{(\neg\neg\bot \supset \bot) \supset ((\neg\neg\bot \supset \bot) \supset (\neg\neg\bot \vee \neg\neg\bot \supset \bot))\} \\
3'. & ((c_2 \cdot c_1) \cdot c_1) : \{\neg\neg\bot \vee \neg\neg\bot \supset \bot\} \\
4'. & c_3 : \{(\neg\neg\bot \vee \neg\neg\bot \supset \bot) \supset \neg(\neg\neg\bot \vee \neg\neg\bot)\} \\
5'. & (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)) : \neg(\neg\neg\bot \vee \neg\neg\bot) \\
6'. & c_4 : \big\{\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))\big\} \\
7. & (\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg \underbrace{(\neg\neg\bot \vee \neg\neg\bot)}_{(1,1)^F}
\end{array}$$

Application: LP Strategies for Nim
Internalization example

1st has a winning strategy in Nim on $(1,2)$ $\big[= (1,2,0)\big]$.

$$(1,2)^T = \neg(0,2)^F \vee \neg(1,0)^F \vee \neg(1,1)^F$$

$1'.$ $c_1 : (\neg\neg\bot \supset \bot)$
$2'.$ $c_2 : \{(\neg\neg\bot \supset \bot) \supset ((\neg\neg\bot \supset \bot) \supset (\neg\neg\bot \vee \neg\neg\bot \supset \bot))\}$
$3'.$ $((c_2 \cdot c_1) \cdot c_1) : \{\neg\neg\bot \vee \neg\neg\bot \supset \bot\}$
$4'.$ $c_3 : \{(\neg\neg\bot \vee \neg\neg\bot \supset \bot) \supset \neg(\neg\neg\bot \vee \neg\neg\bot)\}$
$5'.$ $(c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)) : \neg(\neg\neg\bot \vee \neg\neg\bot)$
$6'.$ $c_4 : \{\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))\}$
$7'.$ $(c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))) : \{(\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg\underbrace{(\neg\neg\bot \vee \neg\neg\bot)}_{(1,1)^F}\}$

Application: LP Strategies for Nim
Internalization example

1st has a winning strategy in Nim on $(1,2)$ $\big[=(1,2,0)\big]$.

$$(1,2)^T = \neg(0,2)^F \vee \neg(1,0)^F \vee \neg(1,1)^F$$

1′. $c_1 : (\neg\neg\bot \supset \bot)$
2′. $c_2 : \{(\neg\neg\bot \supset \bot) \supset ((\neg\neg\bot \supset \bot) \supset (\neg\neg\bot \vee \neg\neg\bot \supset \bot))\}$
3′. $((c_2 \cdot c_1) \cdot c_1) : \{\neg\neg\bot \vee \neg\neg\bot \supset \bot\}$
4′. $c_3 : \{(\neg\neg\bot \vee \neg\neg\bot \supset \bot) \supset \neg(\neg\neg\bot \vee \neg\neg\bot)\}$
5′. $(c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)) : \neg(\neg\neg\bot \vee \neg\neg\bot)$
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Application: LP Strategies for Nim
Internalization example

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Application: LP Strategies for Nim

Computing the explicit strategy

- $c_4 : \{\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))\}$

Application: LP Strategies for Nim

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  c_4: "right, right, continue, continue"

Application: LP Strategies for Nim

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c_4: "right, right, continue, continue"

. . . Win!

Application: LP Strategies for Nim

Computing the explicit strategy

- $c_4 : \{\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))\}$

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Application: LP Strategies for Nim
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  c_4: "right, right, continue, continue"

- Want the strategy $c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$.

Strategy $u \cdot v$: "if $u : (A \supset B)$ and $v : A$, then follow u on B."

Application: LP Strategies for Nim

Computing the explicit strategy

- $c_4 : \{\neg(\neg\neg\bot \vee \neg\neg\bot) \supset ((\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(\neg\neg\bot \vee \neg\neg\bot))\}$

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Computing the explicit strategy

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$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$: "right, continue, continue"

Application: LP Strategies for Nim
Extracting the Nim strategy

$$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)): \text{ “right, continue, continue”}$$

Nim	**Verification**
1st on $(1,2)$ [Pick $((0,2),(1,0))$, or $(1,1)$.]	True on $(1,2)^T$ $[(\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(1,1)^F]$

Application: LP Strategies for Nim
Extracting the Nim strategy

$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$: "right, continue, continue"

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Application: LP Strategies for Nim
Extracting the Nim strategy

$$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))\text{: “right, continue, continue”}$$

Nim	**Verification**
1st on $(1,2)$ [Pick $((0,2)$, $(1,0))$, or $(1,1)$.]	True on $(1,2)^T$ $[(\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(1,1)^F]$
(1st waits.)	True on $\neg(1,1)^F$.

Application: LP Strategies for Nim
Extracting the Nim strategy

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Application: LP Strategies for Nim

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(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$

Application: LP Strategies for Nim

Extracting the Nim strategy

$$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1)) \text{: "right, continue, continue"}$$

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(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$

Application: LP Strategies for Nim
Extracting the Nim strategy

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2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$
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Application: LP Strategies for Nim
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(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$
(2nd waits.)	False on $\neg(1,0)^T$
1st on $(1,0)$. [Pick $(0,0)$.]	True on $(1,0)^T$. $[\neg(0,0) = \neg\bot]$

Application: LP Strategies for Nim
Extracting the Nim strategy

$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$: "right, continue, continue"

Nim	Verification
1st on $(1,2)$ [Pick $((0,2)$, $(1,0))$, or $(1,1)$.]	True on $(1,2)^T$ $[(\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(1,1)^F]$
(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$
(2nd waits.)	False on $\neg(1,0)^T$
1st on $(1,0)$. [Pick $(0,0)$.]	True on $(1,0)^T$. $[\neg(0,0) = \neg\bot]$

Application: LP Strategies for Nim

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(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$
(2nd waits.)	False on $\neg(1,0)^T$
1st on $(1,0)$. [Pick $(0,0)$.]	True on $(1,0)^T$. $[\neg(0,0) = \neg\bot]$
2nd on $(0,0)$.	False on $\bot$.

Application: LP Strategies for Nim
Extracting the Nim strategy

$$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))\text{: “right, continue, continue”}$$

Nim	**Verification**
1st on $(1,2)$ [Pick $((0,2),\ (1,0))$, or $(1,1)$.]	True on $(1,2)^T$ $[(\neg(0,2)^F \vee \neg(1,0)^F) \vee \neg(1,1)^F]$
(1st waits.)	True on $\neg(1,1)^F$.
2nd on $(1,1)$. [Pick $(0,1)$ or $(1,0)$.]	False on $(1,1)^F$. $[\neg(0,1)^T \vee \neg(1,0)^T]$
(2nd waits.)	False on $\neg(1,0)^T$
1st on $(1,0)$. [Pick $(0,0)$.]	True on $(1,0)^T$. $[\neg(0,0) = \neg\bot]$
2nd on $(0,0)$. 1st wins.	False on $\bot$. True wins.

Application: LP Strategies for Nim
Extracting the Nim strategy

$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$: "right, continue, continue"

Nim	**Verification**
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(1st waits.)	True on $\neg(1,1)^F$.
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(2nd waits.)	False on $\neg(1,0)^T$
1st on $(1,0)$. [Pick $(0,0)$.]	True on $(1,0)^T$. $[\neg(0,0) = \neg\bot]$
2nd on $(0,0)$.	False on $\bot$.

Application: LP Strategies for Nim
Extracting the Nim strategy

$c_4 \cdot (c_3 \cdot ((c_2 \cdot c_1) \cdot c_1))$: "right, continue, continue"

Nim strategy on $(1, 2)$.
"take from right, (wait for response), take remaining stick"

Fin

Thanks!

Bryan Renne

http://bryan.renne.org/