# Quantification over Explicit Evidence

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# LP

A quick reminder,
to get the basics in mind.

# Formulas and Axioms

Due to Sergei Artemov

# Proof Polynomials

- variables, $x$, $y$, $\ldots$, intended to range over proofs

- constants, $a$, $b$, $\ldots$, for axioms (more generally, for obvious facts)

- application, if $s$ proves $A \supset B$ and $t$ proves $A$ then $s \cdot t$ proves $B$

- proof checker, if $t$ proves $A$ then $!t$ proves that $t$ proves $A$

- union, $+$ joins two proofs together

# Formulas

- Propositional letters are formulas
- 'Falsehood' is a formula
- If $X$ is a formula and $t$ is a proof polynomial then $t{:}X$ is a formula
- Build up using implication (with other connectives defined, say)

# Axioms

Axioms:

- all classical tautologies
- $t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
- $t{:}X \supset X$
- $t{:}X \supset !t{:}t{:}X$
- $s{:}X \supset (s + t){:}X \qquad t{:}X \supset (s + t){:}X$

Rules

- Modus Ponens
- $c{:}X$ where $X$ is an axiom and $c$ is a proof constant

# *LP* and *S4*

*LP* proofs become *S4* proofs when one replaces every proof polynomial by $\Box$.

*S4* theorems convert into *LP* theorems under some replacement of $\Box$ with proof polynomials.

Realization Theorem

# LP Semantics

Kripke-style

Fitting (2004),

based on an earlier semantics

Mkrtychev (1997).

# Frames

Standard S4 frames, transitive and reflexive.

# Possible Evidence

A proof polynomial counts as *possible evidence* for some formulas, but not for others, at states.

Possible evidence is not certain evidence. Think of it as an expression of *relevance*.

# Evidence Conditions

Closure conditions on proof polynomials.

I'll give the details later, when QLP is presented.

# Weak Models

Truth at worlds is like in S4, except:

$\mathcal{M}, \Gamma \Vdash (t{:}X)$ iff $t$ is evidence for $X$ at $\Gamma$,
and $\mathcal{M}, \Delta \Vdash X$ for every $\Delta$ accessible from $\Gamma$.

This amounts to *Justified true belief.*

# Key Condition

The condition intuitively says that
$t{:}X$ is true at a state provided
$X$ is potentially known,
and $t$ serves as possible evidence for $X$
at that state.

# Fully Explanatory

A weak LP model $\mathcal{M}$ is *Fully Explanatory* provided that, whenever $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$, then for some proof polynomial $t$ we have $\mathcal{M}, \Gamma \Vdash (t{:}X)$.

Intuitively, if $X$ is knowable at a state, there is a reason for $X$.

If $\mathcal{M}$ is a weak LP model, and if the Fully Explanatory condition is also met, then $\mathcal{M}$ is a *strong LP model*.

# Strong Completeness

If the constant specification provides constants for exactly the axioms:

$X$ is valid in all weak models
iff
$X$ is valid in all strong models
iff
$X$ has an axiomatic proof

# QLP

Quantified Logic of Proofs

If $\Box$ is something like
there is a proof of X

why not permit quantification
and make this explicit?

(Fitting, work in progress)

# Syntax

Allow quantification, $(\forall x)X$
where $x$ is a proof variable.

Add new proof polynomial, $(t \forall x)$,
intended to justify $(\forall x)X$ if
$t$ justifies instances of $X$.

Replace proof constants by *primitive proof terms* $f(x_1, \ldots, x_n)$, variables allowed.

If $f(x)$ is a possible reason for $\varphi(x)$ then
$f(t)$ is a possible reason for $\varphi(t)$.

# Axioms

The usual:

1. Classical tautologies.
2. $t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
3. $t{:}X \supset X$
4. $t{:}X \supset\ !t{:}(t{:}X)$
5. $s{:}X \supset (s+t){:}X$ and $t{:}X \supset (s+t){:}X$

# More Axioms

More of the usual:

6. $(\forall x)\varphi(x) \supset \varphi(t)$, for any proof term $t$ that is free for $x$ in $\varphi(x)$.
7. $(\forall x)(\psi \supset \varphi(x)) \supset (\psi \supset (\forall x)\varphi(x))$, where $x$ does not occur free in $\psi$.

And a *uniformity formula*. Assume $y$ does not occur free in $t$ or $\varphi$.

8. $(\exists y)y{:}(\forall x)t{:}\varphi \supset (t \forall x){:}(\forall x)\varphi$.

To conclude we have a proof of $(\forall x)\varphi(x)$, it is not enough to have a proof of each instance, expressed by $(\forall x)(\exists y)y{:}\varphi(x)$.

We want that each instance of $\varphi(x)$ has a *uniform* proof, $(\forall x)t{:}\varphi(x)$.

And we should have a proof of this, $(\exists y)y{:}(\forall x)t{:}\varphi$.

Then conclude we have a proof of $(\forall x)\varphi(x)$, which we can calculate from the uniform proof of instances of $\varphi(x)$, $(t \forall x){:}(\forall x)\varphi$.

$$(\exists y)y{:}(\forall x)t{:}\varphi \supset (t \forall x){:}(\forall x)\varphi$$

# Rules

**Modus Ponens**

$$\frac{X, X \supset Y}{Y}$$

**Axiom Necessitation** if $X$ is an axiom and $p$ is a primitive proof term

$$\frac{}{p{:}X}$$

**Justified Universal Generalization**

$$\frac{t{:}\varphi(x)}{(t \forall x){:}(\forall x)\varphi(x)}$$

# Internalization

**Derived Inference Rule:**

$$\frac{X}{p{:}X}$$

For some proof polynomial $p$

In fact, $p$ ‘reflects’ the proof of $X$

The proof is by induction on proof length.

It is an easy extension of a similar result for LP.

# Universal Generalization

## (Derived Rule)

$\varphi(x)$ (assumed provable)

$p{:}\varphi(x)$ (for some $p$)

$(p \,\forall\, x){:}(\forall x)\varphi(x)$ (justified U. G.)

$(p \,\forall\, x){:}(\forall x)\varphi(x) \supset (\forall x)\varphi(x)$ (axiom 3)

$(\forall x)\varphi(x)$ (modus ponens)

# A Non-Theorem

$$(\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

This is actually a version of an omega rule.

Proof of non-theoremhood later.

# And a Theorem

$$t{:}(\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

and hence

$$(\exists z)z{:}(\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

# Proof

$y{:}\varphi(x) \supset \varphi(x)$ (axiom 3)

$(\exists y)y{:}\varphi(x) \supset \varphi(x)$ (standard quantifier stuff)

$(\forall x)(\exists y)y{:}\varphi(x) \supset (\forall x)\varphi(x)$ (more of the same)

$p{:}[(\forall x)(\exists y)y{:}\varphi(x) \supset (\forall x)\varphi(x)]$ (derived rule)

$p{:}[(\forall x)(\exists y)y{:}\varphi(x) \supset (\forall x)\varphi(x)] \supset$
$[t{:}(\forall x)(\exists y)y{:}\varphi(x) \supset (p \cdot t){:}(\forall x)\varphi(x)]$ (ax. 1)

$t{:}(\forall x)(\exists y)y{:}\varphi(x) \supset (p \cdot t){:}(\forall x)\varphi(x)$ (M. P.)

$t{:}(\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$

# Semantics

Start with a frame, as with LP.

$$\langle \mathcal{G}, \mathcal{R} \rangle$$

States

Accessibility:
reflexive,
transitive

# More Abstract than LP

*Domain function* $\mathcal{D}$
maps members of $\mathcal{G}$ to non-empty sets, *reasons*.

Reasons depend on the state.

Domain functions are *monotonic*,
$\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$.
Reasons are not tenuous.

$\overline{\mathcal{D}}$ is $\cup_{\Gamma \in \mathcal{G}} \mathcal{D}(\Gamma)$ (the *frame domain*)

# Interpretations

$\mathcal{I}$ assigns to $n$-place primitive function symbol $f$ an $n$-place function $f^{\mathcal{I}} : \overline{\mathcal{D}}^n \longrightarrow \overline{\mathcal{D}}$.

$\mathcal{I}$ assigns to the one-place function symbol ! a mapping $!^{\mathcal{I}} : \overline{\mathcal{D}} \longrightarrow \overline{\mathcal{D}}$.

$\mathcal{I}$ assigns to the two-place function symbol $\cdot$ a binary operation $\cdot^{\mathcal{I}} : \overline{\mathcal{D}} \times \overline{\mathcal{D}} \longrightarrow \overline{\mathcal{D}}$,
and similarly for $+$.

# And for quantification

$\mathcal{I}$ assigns to $\forall$ a mapping $\forall^{\mathcal{I}}$ from the function space of $\overline{\mathcal{D}}$ to $\overline{\mathcal{D}}$ itself.

$$\forall^{\mathcal{I}} : (\overline{\mathcal{D}} \longrightarrow \overline{\mathcal{D}}) \longrightarrow \overline{\mathcal{D}}$$

The rough idea: if $\varphi(x)$ is a formula
and $f(x)$ provides a reason for $\varphi(x)$
'uniformly' for each $x$,
then $\forall^{\mathcal{I}}(f)$ is a reason for $(\forall x)\varphi(x)$.

# Closure Conditions

For each $\Gamma \in \mathcal{G}$, $\mathcal{D}(\Gamma)$ is closed under:

- $f^{\mathcal{I}}$ for every primitive function symbol $f$
- $.^{\mathcal{I}}$
- $!^{\mathcal{I}}$
- $+^{\mathcal{I}}$

# Valuations

A map $v$ from proof variables to members of $\overline{\mathcal{D}}$.

Each proof term $t$ maps to a member of $\overline{\mathcal{D}}$ by:

- $x^v = v(x)$ for $x$ a variable
- $f(t_1, \ldots, t_n)^v = f^{\mathcal{I}}(t_1^v, \ldots, t_n^v)$ for $f$ a primitive function symbol
- $(t \cdot u)^v = (t^v \cdot^{\mathcal{I}} u^v)$
- $(t + u)^v = (t^v +^{\mathcal{I}} u^v)$
- $(!t)^v = !^{\mathcal{I}}(t^v)$

# More on Valuations

- Suppose $t^w$ has been defined.
  For each variable $x$, define a mapping, $(x \to t^v) : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$ as follows. For each $r \in \overline{\mathcal{D}}$, $(x \to t^v)(r)$ is $t^w$ where $w = v\binom{x}{r}$.

Suppose that for every proof variable $y$ occurring in $t$, other than $x$, $v(y) \in \mathcal{D}(\Gamma)$. Then it is required that $(t \forall x)^v \in \mathcal{D}(\Gamma)$.

# Evidence Functions

An *evidence function* $\mathcal{E}$ is a map that assigns to each $\Gamma \in \mathcal{G}$, to each $r \in \overline{\mathcal{D}}$, and to each valuation $v$, a set $\mathcal{E}(\Gamma, r, v)$ of formulas.

Think of the members of $\mathcal{E}(\Gamma, r, v)$ as the formulas that $r$ provides possible evidence for, in state $\Gamma$, using $v$ to supply values for the free variables of the formulas.

# Evidence Conditions

- If $v$ and $w$ agree on the free variables of $X$, then $X \in \mathcal{E}(\Gamma, r, v)$ iff $X \in \mathcal{E}(\Gamma, r, w)$

# Truth Definition

- $\mathcal{M}, \Gamma \Vdash_v P \iff \Gamma \in \mathcal{V}(P)$
  for propositional $P$ (arbitrary specification)

- $\mathcal{M}, \Gamma \nVdash_v \bot$

- $\mathcal{M}, \Gamma \Vdash_v X \supset Y \iff$
  $\mathcal{M}, \Gamma \nVdash_v X$ or $\mathcal{M}, \Gamma \Vdash_v Y$

- $\mathcal{M}, \Gamma \Vdash_v (\forall x)\varphi \iff \mathcal{M}, \Gamma \Vdash_w \varphi$ for every $w$ where $w = v\binom{x}{r}$ and $r \in \mathcal{D}(\Gamma)$

- $\mathcal{M}, \Gamma \Vdash_v (t{:}X) \iff X \in \mathcal{E}(\Gamma, t^v, v)$ and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$

# Models

What was just defined is a *weak* model.

A *strong* model is also ***fully explanatory***, where:

if $X$ is meaningful at state $\Gamma$, and $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, then $X \in \mathcal{E}(\Gamma, r, v)$, for some $r \in \mathcal{D}(\Gamma)$.

Informally, this says that if $X$ is known in the Hintikka sense, then $X$ has a reason.

# An Example

Here is an example of a QLP model.

I'm interested in a formula that does not involve +, !, etc., so I won't specify an interpretation.

And I'll be somewhat informal about valuations. I'll use members of the model domain as if they were constants in the language.

$$\begin{array}{c} \Gamma \quad \boxed{1} \quad \Vdash P \\ \downarrow \\ \Delta \quad \boxed{1,2} \quad \Vdash P \end{array}$$

$$\mathcal{E}(\Gamma, 1, v) = \mathcal{E}(\Delta, 1, v) = \text{all formulas}$$

$$\mathcal{E}(\Gamma, 2, v) = \mathcal{E}(\Delta, 2, v) = \emptyset$$

$P \in \mathcal{E}(\Gamma, 1, v)$ and $P \in \mathcal{E}(\Delta, 1, v)$

$\Gamma \Vdash P$ and $\Delta \Vdash P$

so

$\Gamma \Vdash 1{:}P$ and $\Delta \Vdash 1{:}P$

$1{:}1{:}P \in \mathcal{E}(\Gamma, 1, v)$ so $\Gamma \Vdash 1{:}1{:}P$

$\mathcal{D}(\Gamma) = \{1\}$ so $\Gamma \Vdash (\exists y) y{:}1{:}P$

$\mathcal{D}(\Gamma) = \{1\}$ so $\Gamma \Vdash (\forall x)(\exists y) y{:}x{:}P$

$P \notin \mathcal{E}(\Delta, 2, v)$ so $\Delta \not\Vdash 2{:}P$

$\mathcal{D}(\Delta) = \{1, 2\}$ so $\Delta \not\Vdash (\forall x)x{:}P$

so $\Gamma \not\Vdash 1{:}(\forall x)x{:}P$

$\mathcal{D}(\Gamma) = \{1\}$ so $\Gamma \not\Vdash (\exists y)y{:}(\forall x)x{:}P$

$$\Gamma \not\Vdash (\forall x)(\exists y)y{:}x{:}P \supset (\exists y)y{:}(\forall x)x{:}P$$

or, taking $x{:}P$ to be $\varphi(x)$,

$$\Gamma \not\Vdash (\forall x)(\exists y)y{:}\varphi(x) \supset (\exists y)y{:}(\forall x)\varphi(x)$$

which was a formula that was mentioned earlier.

# Results

Just a sketch.

Proofs omitted.

# Soundness and Completeness

Provablity coincides with
validity in weak models
and with
validity in strong models.

Completeness is by a kind of Henkin argument
giving a ‘canonical’ term model.

# Conservativeness

QLP conservatively extends LP.

The proof is largely semantic.

Of course some adjustment must be made
for the shift from proof constants
to more complex
primitive proof terms.

Not a big deal.

# Embedding

S4 embeds in QLP, translating "necessity" by "there exists a proof of..."

Also primarily a semantic proof.

# For Example

$\Box P \supset \Box\Box P$

becomes the QLP theorem

$(\exists x)x{:}P \supset (\exists x)x{:}(\exists y)y{:}P$

Of course QLP has theorems that are not translates of LP theorems.

# Hopes for the Future

Combine QLP with a multi-agent logic of knowledge, Hintikka-style.

Introduce an operator $K_i$ for each 'knower'.

But still allow reasons, with quantifiers over them.

We could distinguish between agent $i$ knowing $X$ has a reason, $K_i(\exists x)x{:}X$, and agent $i$ having a reason for $X$, $(\exists x)K_i x{:}X$

We could say agent $i$ has a reason for $X$, and $j$ knows this, without having to say that $j$ knows what the reason is, $(\exists x)K_i x{:}X \wedge K_j(\exists x)K_i x{:}X$

We could say agent $j$ knows there is a reason for $X$, but does not know what it is, $K_j(\exists x)x{:}X \wedge \neg(\exists x)K_j x{:}X$

This assumes a common domain of reasons, for all agents.

Perhaps we also want to consider separate domains.

Or maybe that just gets too complex?

"The awful thing about life is that everyone has their reasons."

# But

That is for future work.

Thank you.