Intuitionistic Finitely Many-Valued Sequent Calculi

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Abstract

We present an intuitionistic sequent calculus for arbitrary finitely many-valued logics which has a positive semantics given by a natural extension of Kripke models. Our calculus has two types of rules: the *introduction* rules and the *pseudo-cut* rules. The introduction rules introduce formulas at all places on the left or on the right hand side of a sequent. They generalize the introduction rules of 2-places sequent calculus. The pseudo-cut rules allow to introduce formulas in the middle of a sequent. They may not exist in a 2-places sequent calculus. We show the soundness and completeness of our system and prove the cut-elimination theorem.

Keywords: finitely many-valued logics, sequent calculus, Kripke models, cut elimination

1 Introduction

This paper presents an intuitionistic sequent calculus for many-valued propositional logics. The study of such calculi is interesting from two points of view : first we want to get a general framework for an intuitionistic sequent calculus that would be sound and complete for some extension to m-valued case of Kripke models [4]. Such a framework helps answering the following question : among the ways allowing to obtain an intuitionistic system (like LJ [3]) from a classical one (like LK [3]) which one can easily be generalized to a m-valued case. Second, does a m-valued intuitionistic calculus have a nice computational interpretation ?
The intuitionistic many-places sequent calculi were already studied in [6] and [2]. In both cases a natural extension of Kripke models to the many-valued logics was proposed and a sound and complete sequent calculus was given for such models. However in both cases some additional restrictions were needed. In [2] the semantics used is the negative one, i.e. a sequent is satisfied by an interpretation iff for some place i there is a formula at this place that does not take the value v_i. In [6] to get a nice semantics the sequents are restricted to a special syntactical form where one formula can occur only at all places on the left or on the right hand side of the sequent, which means no "isolated" formula in the middle of a sequent. The introduction rules introduce formulas on every place left from or on every place right from any given place i and thus no introduction is possible in the middle of a sequent.
In this work we propose sequent calculi (with positive semantics i.e. where a sequent is satisfied iff there is a formula at a place i which takes the corresponding value v_i) that are sound and complete for the extension of Kripke models proposed in [6] and

which use the most general form of many-places sequents. These calculi are obtained from the intuitionistic calculi of [6] by adding a new rule called *pseudo-cut* which is the only one allowing to introduce formulas in the middle of a sequent. Such a rule may only exist for a m-places sequent calculus where $m > 2$. We prove the soundness and the completeness of these calculi and we show that they have the subformula property. Then we consider intuitionistic sequent calculi extended with a cut-rule and we prove the cut-elimination theorem for them.
Finally we study the following question : may the pseudo-cut rule be replaced by a single rule of introduction in the middle of a sequent ? In the classical sequent calculus the proofs with pseudo-cuts may be transformed to a special form where the active formulas of the pseudo-cuts are the conclusions of the left and right introductions. Thus the pseudo-cut rule may be replaced by a "direct" introduction of the formula in the middle of the sequent and we get a sequent calculus as in [1]. We show that in the intuitionistic case the use of pseudo-cuts may not be reduced to this special form, that would "emulate" a single rule of introduction in the middle of a sequent.

2 A generic intuitionistic sequent calculus

We describe here a generic intuitionistic sequent calculus system. The system is generic because it is designed for any m-valued propositional logic (with $m \geq 3$). It should be seen as a framework from which calculi for particular logics can be built. Let us call $\mathcal{V}$ the finite set $\{v_1; \ldots ; v_m\}$ of all truth values.

2.1 Formulas and sequents

We consider many-places sequents as those presented in [5], [1].
The set $\mathcal{F}$ of propositional formulas is defined by induction both from an enumerable set $\mathcal{A}$ of atomic formulas (also called propositional variables) $a_1, \ldots , a_n, \ldots$ and from a finite set C of connectives.
Let c^r be any connective with arity r.
A sequent is a list of m *multi-sets* of formulas $\Gamma_1| \ldots |\Gamma_i| \ldots |\Gamma_m$. The positions $1 \ldots m$ in this list are called the *places* of the sequent.
Note that the sequents are not restricted to a special form (as opposed to Rousseau's sequents [6]): a formula may occur in any set of places in a sequent.
Some notations and definitions:
Upper case Greek letters $\Gamma, \Delta, \Sigma, \Pi, \Omega, \ldots$ are used to denote sequents and if Γ is a sequent then $\Gamma_1, \ldots , \Gamma_m$ are the m multi-sets which compose it, thus $\Gamma = \Gamma_1| \ldots |\Gamma_m$.
By Γ_i, A we denote the multi-set of formulas obtained by adding an occurrence of A in Γ_i. By A^* we denote any number (but at least one) of occurrences of the formula A.
Γ_i, Δ_j is the *union* of the multi-sets Γ_i and Δ_j.
If Γ and Π are both sequents then $\Gamma\Pi$ is their *union* which is the sequent where an occurrence of a formula A at any place i comes either from an occurrence of A in Γ or an occurrence of A in Π at the same place. $(\Gamma)^n$ denotes the union of n occurrences of the sequent Γ.
For any multi-set of formulas Γ_0 and any set of places I ($I \subseteq \{1; \ldots ; m\}$) we use $|\Gamma_0|_I$ to denote the subsequent of Γ_0 whose occurrences of formulas are exactly those of Γ_0

at the places I. If the places in I are consecutive we use $|\Gamma_0|_{i,j}$ instead of $|\Gamma_0|_{\{i;\dots;j\}}$. If I has only one element i, we abbreviate $|\Gamma_0|_{\{i\}}$ by $|\Gamma_0|_i$.

2.2 Deduction rules

The deduction rules are divided in four sets: axioms, introductions (*left introductions* and *right introductions*), structural rules and *pseudo-cuts*. We shall see that no rules but the *pseudo-cuts* may introduce a formula only in the "middle" of a sequent.
There is a *left introduction* rule (i.e. to all places $\leq i$) for every place $1 \leq i < m$ and a *right introduction* rule (i.e. to all places $\geq i$) for every place $1 < i \leq m$ and for every connective c^r.
These introduction rules are generated in a mechanical way from the truth table $f_c : \mathcal{V}^r \to \mathcal{V}$ of the connective c^r through the conjunctive normal forms ([5], [1]) of $f_c(x_1, \dots, x_r) \leq v_i$ (for the left introduction rule to the places $\leq i$) and of $f_c(x_1, \dots, x_r) \geq v_i$ (for the right introduction rule to the places $\geq i$).
Contractions and *weakenings* of formulas are done explicitly by using the structural rules.
We shall remark that our introduction and structural rules are the same as those of Rousseau's intuitionistic sequent calculus [6].
The sequent calculus is given by the following rules:

- Axioms are of the form $A|\dots|A$.
- For any connective c^r and any truth value v_i there is a left introduction to all places from 1 to i:

  $$\frac{\Gamma^j\ \Pi^j(A_1, \dots, A_r)\ (1 \leq j \leq n)}{\Gamma\ |c(A_1, \dots, A_r)|_{1,i}}\ c\,Int : 1, \dots, i$$

  Where:
  n is the number of premises of the rule
  Every sequent $\Pi^j(A_1, \dots, A_r)$ is the *active* subsequent of the j^{th} premise.
  Γ is the union of all the contexts of the premises: $\Gamma = \Gamma^1 \cup \dots \cup \Gamma^j \cup \dots \cup \Gamma^n$.
- For any connective c^r and any truth value v_i there is a right introduction to all places from i to m:

  $$\frac{\Gamma^j\ \Pi^j(A_1, \dots, A_r)\ (1 \leq j \leq n')}{\Gamma\ |c(A_1, \dots, A_r)|_{i,m}}\ c\,Int : i, \dots, m$$

  With the same conventions of notations as for the left introductions but with the following syntactical restriction on the context $\Gamma^j = \Gamma^j_1|\dots|\Gamma^j_m$ of each premise:
  (RC)

  $$\mathcal{F}(\Gamma^j_m) \subseteq \dots \subseteq \mathcal{F}(\Gamma^j_i) \subseteq \mathcal{F}(\Gamma^j_{i-1}) \subseteq \dots \subseteq \mathcal{F}(\Gamma^j_1)$$

  Where $\mathcal{F}(\Gamma^j_i)$ denotes the set of formulas of Γ^j_i.
  As in [6] this restriction is the only syntactical difference between the intuitionistic and the classical system.

- For any place i there is a weakening at the place i:

$$\frac{\Gamma_1|\ldots|\Gamma_i|\ldots|\Gamma_m}{\Gamma_1|\ldots|\Gamma_i, A|\ldots|\Gamma_m} W:i$$

- For any place i there is a contraction at the place i:

$$\frac{\Gamma\,|A, A|_i}{\Gamma\,|A|_i} C:i$$

- For any place i (with $1 < i < m$) there is a *pseudo-cut* at the place i:

$$\frac{\Gamma\,|A|_{1,i} \quad \Delta\,|A|_{i,m}}{\Gamma\Delta\,|A|_i} \,pcut:i$$

  The pseudo-cut rule may only exist in a many-valued logic and is the only rule which allows the introduction of formulas in the middle of a sequent. It looks like the well-known cut rule of the two-valued sequent calculi, but the formula A can not completely disappear from the conclusion. As we shall see in the section **4.2** the *pseudo-cuts* are needed for the completeness of the system.

The system obviously has the *subformula property*, which we state as follows :

PROPOSITION 2.1
If a m-places sequent Γ is provable in our system then in any proof p of Γ all formulas are subformulas of formulas of Γ.

3 Semantics

We recall here the extension of Kripke's semantics to many-valued logics used by Rousseau in [6].
A Kripke model $\mathcal{K}$ for an m-valued logic is a structure $< K, \leq, Val >$ where :

- K is a set of nodes
- $\leq$ is a pre-order over K
- $Val = (v_k)_{k\in K}$ is a set of *valuations* $v_k : \mathcal{A} \to \mathcal{V}$ of atomic formulas such that :
  $\forall k, k' \in K$ if $k \leq k'$ then $\forall a \in \mathcal{A}\ v_k(a) \leq v_{k'}(a)$

Any valuation v_k for a node k of a model $\mathcal{K}$ is extended to an *interpretation* $v_k^* : \mathcal{F} \to \mathcal{V}$ of all formulas, by induction on the construction of formulas:

- $v_k^*(a) = v_k(a)$ for any atomic formula a
- $v_k^*(c(A_1, \ldots, A_r)) = inf_{k'\geq k}(f_c(v_{k'}^*(A_1), \ldots, f_c(v_{k'}^*(A_r))$ for any connective c^r.
  Where $f_c : \mathcal{V}^r \to \mathcal{V}$ is the truth table of c.

For any truth value $v_i \in \mathcal{V}$ we write $k \Vdash F^{v_i}$ iff $v^*(F) = v_i$, which means that F takes the value v_i at the node k.

The following property expresses that the interpretation of formulas is growing through the nodes of a model.

PROPOSITION 3.1
$\forall k, k' \in K$ if $k \leq k'$ then $\forall F \in \mathcal{F}\ v_k^*(F) \leq v_{k'}^*(F)$

3.1 Interpretation of a sequent in a Kripke model

The last step is to define how sequents are interpreted.
Let K be a m-valued Kripke model. For any m-places sequent Π we write $k \Vdash \Pi$ iff Π is *satisfied* by the node k of K, which means that there exists a place $1 \leq i \leq m$ and an occurrence of a formula at the place i such that $k \Vdash F^{v_i}$. We shall say that this occurrence of F is *satisfied* by the node k.
We write $\mathcal{K} \Vdash \Pi$ (i.e. Π is true in the model $\mathcal{K}$) iff $k \Vdash \Pi$ for every node $k \in K$.

4 Soundness and completeness

We now prove that our sequent calculus is *sound* and *complete* with respect to this semantics. These are properties needed for any proof theoretic system.

4.1 Soundness

The soundness statement expresses that all provable sequents are true, where the meaning of "true" depends on the chosen semantics.

THEOREM 4.1
If a m-places sequent Π is provable then it is true in all m-valued Kripke models.

PROOF. The proof is a straightforward adaptation from [6] and thus will not be detailed here. Note that the restriction (RC) is needed in the case of right introductions. ∎

4.2 Completeness

THEOREM 4.2
Every sequent Π which is true in all m-valued Kripke models is provable.

PROOF. Let us take any non provable sequent Π and find a Kripke model in which Π is false.
We proceed in two steps : first we add formulas to Π in order to get a non provable sequent in the form of Rousseau's sequents [6], then we apply the completeness theorem of Rousseau's sequent calculus [6].

1. We first transform the sequent Π in the following way :
   Let us call an *isolated* formula any formula which is neither at all places left from nor at all places right from a place i $(1 < i < m)$.
   We choose an isolated formula A in Π (if there is any).
   Let $|A|_{<i}$ be the sequent $A|\dots|A|\dots$ where A is at all places $< i$ and let $|A|_{>i}$ be the sequent $\dots|A|\dots|A$ where A is at all places $> i$.
   As we suppose that Π is not provable, we know that sequent unions $\Pi|A|_{<i}$ and $\Pi|A|_{>i}$ may not be both provable, because otherwise we would prove Π by applying a pseudo-cut rule.

   We then define a sequent Π^1 from Π as :
   $\Pi^1 = \Pi|A|_{<i}$ if $\Pi|A|_{<i}$ is not provable
   $\quad = \Pi|A|_{>i}$ otherwise.

By repeating this process with an isolated formula in Π_1 and so on, we finally get a finite sequence $\Pi^0, \ldots, \Pi^l$ of sequents such that :

- $\Pi^0 = \Pi$
- None of the sequents Π^j is provable.
- Π^j is a subsequent of Π^{j+1}.
- Π^{j+1} has one less isolated formula than Π^j.
- Π^l does not have any isolated formula.

2. We apply Rousseau's completeness theorem :

   Let $\Pi' = \Pi^l$.

   Π' is not provable and contains Π. As it has no isolated formula, it matches Rousseau's definition of many-places sequents [6].

   As Rousseau's intuitionistic sequent calculus is a subsystem of our calculus, we know that Π' is not provable in Rousseau's calculus.

   By Rousseau's completeness theorem [6] there is a Kripke model $\mathcal{K}_0$ in which Π' is false.

   As Π in included in Π', Π is false in $\mathcal{K}_0$ too.

∎

5 Cut elimination

The second part of this paper deals with the process of *cut elimination*. Proving cut elimination shows that our calculus is *analytic*, because cut-free proofs have the *subformula property*. Furthermore, the cut elimination process is the dynamics of the calculus and should correspond to some kind of computation in an algorithmic interpretation (hypothetical, so far) of our sequent calculus, as in the two values case [7].

5.1 Cuts

We first define a notion of *cut* for our m-places sequents and discuss the difference between cuts and pseudo-cuts.
A cut is a rule of the form :

$$\frac{\Gamma|A|_i \quad \Delta|A|_j}{\Gamma\Delta}\,cut_{i,j}$$

Where i, j are any two different places.

Let us first remark that the system extended with a general cut rule is equivalent to the initial system (**2.2**). However, pseudo-cuts allow the subformula property, but cuts obviously do not.
Second, for the extended system cut-elimination leads to replace some cut rules with pseudo-cuts. Typically a proof of the following form (in the case of 3-places sequents):

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma\, A|A|\_}\,Int:0,* \qquad \dfrac{\vdots}{\Delta\, \_|A|A}\,Int:*,1}{\Gamma\Delta\, \_|A,A|\_}\,cut:0,1}{\Gamma\Delta\, \_|A|\_}\,C:*$$

is transformed as :

$$\dfrac{\dfrac{\vdots}{\Gamma\, A|A|\_}\,Int:0,* \qquad \dfrac{\vdots}{\Delta\, \_|A|A}\,Int:*,1}{\Gamma\Delta\, \_|A|\_}\,pcut:*$$

We shall see in the following sections how the notion of pseudo-cuts is needed to represent rules which "cut formulas" but which can not be eliminated from proofs in general.

Finally, the use of pseudo-cuts may also not be reduced to the simulation of "middle introductions" (i.e. introductions to a place i $(1 < i < m)$), as we shall see in the section **6**.

<u>Generalized cuts</u>:

To simplify the proofs we use in the following section a more general notion of cut, based on disjoint sets of places. It is called *gcut* as "generalized cut" and defined as follows :

$$\dfrac{\Gamma\,|A|_I \quad \Delta\,|A|_J}{\Gamma\Delta}\,gcut:I,J$$

Where I, J are both non empty sets of places and $I \cap J = \emptyset$.

PROPOSITION 5.1
The *gcut* rule is derivable.

PROOF. We just use (usual) cuts and contractions to get the conclusion $\Gamma\Delta$ from $\Gamma\,|A|_I$ and $\Delta\,|A|_J$. The *gcut* is finally obtained through $n \times p$ cuts, where n, p are respectively the number of places in I, J. ■

5.2 Weak normalization

Let MLJ (as "many-places LJ") be the many-places intuitionistic sequent calculus presented in the section **2.2** to which we add all cut rules.
We shall now prove the *weak normalization* property for MLJ, which we state as follows :

THEOREM 5.2
For any proof p in MLJ there is a cut-free proof in MLJ with the same conclusion.

The proof of this result is mainly based on the following lemma, for the case where a cut is done immediately after two introductions (left and right) and the occurrences of formulas which are being cut are taken among the occurrences that were just introduced.

5.3 Cut elimination lemma

LEMMA 5.3
Let p be a proof which ends with a cut preceded by two introduction rules (one to the left and one to the right) such that only introduced occurrences are cut.

$$\dfrac{\dfrac{\begin{matrix}\vdots\, p_1^k\\ \Gamma^k\Pi^k(A_1,\ldots,A_r,B)\,(1\leq k\leq n)\end{matrix}}{\Gamma\,|A|_{1,i}}\,c\,Int:1,\ldots,i \qquad \dfrac{\begin{matrix}\vdots\, p_2^l\\ \Delta^l\Sigma^l(A_1,\ldots,A_r,B)\,(1\leq l\leq n')\end{matrix}}{\Delta\,|A|_{j,m}}\,c\,Int:j,\ldots,m}{\Gamma|A|_{1,i'-1}|A|_{i'+1,i}\Delta|A|_{j,j'-1}|A|_{j'+1,m}}\,cut:i',j'$$

Where c is any connective of arity $r{+}1$ ($r\geq 0$) and the cut formula A is $c(A_1,\ldots,A_r,B)$. If all subproofs above both introduction rules are cut-free then there exists another proof p' with the same conclusion as p where the only cuts are done on the subformulas $A_1,\ldots,A_r,B$ of A.

Proving this lemma requires the following technical property (which is easily proven by applying *gcuts*):

PROPOSITION 5.4
Let Γ be any m-places sequent $\Gamma_1|\ldots|\Gamma_m$ and A be any formula.
Let $\Omega^1,\ldots,\Omega^i,\ldots,\Omega^m$ be the m sequents obtained from Γ by adding an occurrence of A respectively to all places but the place i (with $1\leq i\leq m$).
If $\Omega^1,\ldots,\Omega^m$ are provable then Γ is provable.

We now prove the main lemma **5.3**:

PROOF. To eliminate the cut on A in p we need to use the premises of both introduction rules in order to cut all active occurrences of the subformulas $A_1,\ldots,A_r,B$ of A.
As we keep the most general definition of the introduction rules, we have to invoke here the way these rules are built from the normal forms to state the following properties for the premises of both introductions of A :
For every r-tuple $(i_1,\ldots,i_r)$ of places among $\{1;\ldots;m\}$:

1. There exists exactly one premise $\Gamma^k\Pi^k(A_1,\ldots,A_r,B)$ of the left introduction and exactly one premise $\Delta^l\Sigma^l(A_1,\ldots,A_r,B)$ of the right introduction such that each formula A_s ($1\leq s\leq r$) is at places in $\Pi^k(A_1,\ldots,A_r,B)$ and $\Sigma^l(A_1,\ldots,A_r,B)$ but the place i_s.
   Let us denote respectively by $\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$ and $\Sigma^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$ the active parts of both sequents.
   Let us denote by $\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r)$ the subsequent of both $\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$ and $\Sigma^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$ where B has no occurrence.
   Then we have $\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)=\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r)\,|B|_I$ and $\Sigma^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)=\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r)\,|B|_{I'}$.
   Where I,I' are the sets of places where B has an occurrence in each sequent.
2. There is no place where B occurs both in $\Pi^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$ and $\Sigma^{(i_1,\cdots,i_r)}(A_1,\ldots,A_r,B)$.

Both properties are trivial consequences of the way normal forms are built from the truth tables of each many-valued logic.
Property 1 requires that each introduction of A exactly has m^r premises. This is true if axioms are kept as premises. Thus we have $n = n' = m^r$.
Property 2 is true because both introductions are done on disjoint sets of places (we suppose $i < j$).

By property 2 we can first eliminate all active occurrences of B in the premises of both introductions. In the following we omit contexts in the sequents to simplify notations.
For each r-tuple $(i_1, \ldots, i_r)$ of places, we build a proof of $\Pi^{(i_1,\cdots,i_r)}(A_1, \ldots, A_r)$ by *gcut* on B from the premises $\Pi^{(i_1,\cdots,i_r)}(A_1, \ldots, A_r, B)$ and
$\Sigma^{(i_1,\cdots,i_r)}(A_1, \ldots, A_r, B)$.

Then property 1 and proposition **5.4** may be invoked to cut successively the formulas $A_r, \ldots, A_1$.
Indeed for each $r-1$-uplet $(i_1, \ldots, i_{r-1})$ we got by *gcuts* on B a proof of each of the following sequents:
$\Pi^{(i_1,\cdots,i_{r-1},1)}(A_1, \ldots, A_{r-1}, A_r), \ldots, \Gamma\Pi^{(i_1,\cdots,i_{r-1},m)}(A_1, \ldots, A_{r-1}, A_r)$.
By proposition **5.4** we eliminate by successive gcuts all active occurrences of A_r: thus we prove the m^{r-1} subsequents $Pi^{(i_1,\cdots,i_{r-1})}(A_1, \ldots, A_{r-1})$.
We repeat this process to eliminate successively all active occurrences of $A_{r-1}, \ldots, A_1$: if we add contractions in the context of the conclusion we finally get a proof of $\Gamma\Delta$.■

5.4 Notion of mix

To prove the cut elimination theorem we use the standard method ([3]) which consists in replacing the *cut* rule by the following rule, called *mix*, where all occurrences of A in each of the places i, j are being cut:

$$\frac{\Gamma\,|A^*|_i \quad \Delta\,|A^*|_j}{\Gamma\Delta}\, mix : i, j$$

Where $i \neq j$ and A has no occurrence at the place i in Γ and A has no occurrence at the place j in Δ. This rule is obviously derivable using cuts and contractions: we first contract all occurrences of A in $|A^*|_i$ and $|A^*|_j$, then we apply a *cut* : i, j. We can conversely derive the *cut* rule from the *mix* rule by applying weakenings to the conclusion of the *mix*.

Let us call MLJ' the sequent calculus obtained from MLJ by replacing *cuts* by *mixes*: MLJ and MLJ' are obviously equivalent.
We shall deduce weak normalization for MLJ from weak normalization for MLJ'.

5.5 Weak normalization of MLJ'

The theorem **5.2** is an immediate corollary of the following result for MLJ':

THEOREM 5.5
For any proof p in MLJ' there is a proof p' with the same conclusion which is translated in MLJ as a cut-free proof.

PROOF. Let p be any proof in MLJ'. We denote by $t(p)$ the size of p (i.e. the number of nodes in the proof tree). We call *degree* of a *mix* the length of the cut formula and $d(p)$ denotes the maximal degree of a *mix* in p. Let $n(p)$ be the number of *mixes* which degree is $d(p)$ in p.
The proof is done by induction on the 3-tuple $(d(p), n(p), t(p))$ (with the lexicographic ordering).
All cases on the last rule R of p have to be enumerated, but for lack of space we can not detail them here. ∎

6 A discussion about middle introductions

Let us consider here the particular case of MLJ for the 3-valued Lukasiewicz logic with the set of connectives $\{\neg; \rightarrow\}$. The 3 values are denoted $\{0; *; 1\}$.
We want to show that pseudo-cuts can not be reduced to a kind of "middle introduction" rule, that is to say a rule which would allow to introduce directly a formula in the middle place in the 3-places sequents. As we already remarked it, pseudo-cuts are the only means to prove sequents with "isolated" formulas (which are excluded by Rousseau's definition of sequents [6]).
Nevertheless we prove here that the use of pseudo-cuts can not be restricted to the following case, where the pseudo-cut immediately follows the left and the right introduction of the *pcut*-formula:

$$\dfrac{\dfrac{}{\Gamma_1, A|\Gamma_2, A|\Gamma_3}\,Int^- : 0, * \qquad \dfrac{}{\Delta_1|\Delta_2, A|\Delta_3, A}\,Int^+ : *, 1}{\Gamma_1, \Delta_1|\Gamma_2, \Delta_2, A|\Gamma_3, \Delta_3}\,pcut$$

One may consider this succession of two introductions (respectively to the left and to the right) and a *pcut* as a "macro-rule" of introduction to the place $*$. Let us then call $macro - Int : *$ such a sequence of rules.
Let $MLJL$ (as "MLJ Light") be the subsystem of MLJ where it is forbidden to use a *pcut* anywhere but in a macro-introduction to the place $*$. In the following we prove that $MLJL$ is not complete for Lukasiewicz 3-valued logic, by showing a sequent which is provable in MLJ but not in $MLJL$.

6.1 Introduction rules for Lukasiewicz 3-valued logic

We recall here the introduction rules of MLJ (and thus of $MLJL$) for the connectives $\{\neg; \rightarrow\}$. These rules are built from simplified normal forms (such a simplification is possible too in the general case **2.2** as shown in [1]). Contexts are omitted to simplify notations.

1. Introduction rules for $\neg$:

$$\dfrac{\_|\_|A}{\neg A|\_|\_}\,Int^- : 0 \qquad \dfrac{\_|A|A}{\neg A|\neg A|\_}\,Int^- : 0, *$$

$$\dfrac{A|\_|\_}{\_|\_|\neg A}\,Int^+ : 1 \qquad \dfrac{A|A|\_}{\_|\neg A|\neg A}\,Int^+ : *, 1$$

2. Introduction rules for $\to$:

$$\dfrac{\_|\_|A \quad B|\_|\_}{A \to B|\_|\_}\ Int^- : 0$$

$$\dfrac{\_|A|A \quad B|B|\_ \quad B|\_|A}{A \to B|A \to B|\_}\ Int^- : 0, *$$

$$\dfrac{A|B|B \quad A|A|B}{\_|\_|A \to B}\ Int^+ : 1$$

$$\dfrac{A|A, B|B}{\_|A \to B|A \to B}\ Int^+ : *, 1$$

6.2 Remarkable sequents

Let us now consider 3-places sequents of the following form:
$B^m|(A \to B)^n, A^p|\neg\neg(A \to B)^q, A^r$ (1)
where A, B are atomic formulas and m, n, p, q, r are the number of occurrences of each formula in the sequent. These sequents have the following property:

PROPOSITION 6.1
If Π is a sequent of the form (1) and either $m = 0$ or $n = 0$ or $q = 0$ then Π is not provable in MLJ.

PROOF. In each of these 3 cases we easily find a Kripke model in which Π is false ... ∎

But if none of n, m, q is zero, then these sequents are all provable in MLJ.

PROPOSITION 6.2
Any sequent of the form (1) such that $n \neq 0, m \neq 0$ and $q \neq 0$ is provable in MLJ.

PROOF. We just prove the subsequent $B|(A \to B)|\neg\neg(A \to B)$.

$$\dfrac{\dfrac{\dfrac{A \to B|A \to B|A \to B}{\neg(A \to B), A \to B|A \to B|}\ Int^- : 0}{A \to B|A \to B|\neg\neg(A \to B)}\ Int^+ : 1 \qquad \dfrac{\dfrac{B|B|B}{A, B|A, B|B}\ W : 0, *}{B|A \to B|A \to B}\ Int^+ : *, 1}{B|A \to B|\neg\neg(A \to B)}\ pcut$$ ∎

PROPOSITION 6.3
None of the sequents of the form (1) is provable in $MLJL$.

PROOF. Let Π be a sequent of the form (1). We reason by absurdity by supposing that there is a proof p of Π in $MLJL$.
We enumerate all possible forms for p: let R be the last rule in p which is neither a weakening nor a contraction. By property **6.1** above, we know that the conclusion of R is a sequent of the following form:
$B, B^{m_0}|A \to B, (A \to B)^{n_0}, A^{p_0}|\neg\neg(A \to B), \neg\neg(A \to B)^{q_0}, A^{r_0}$
Moreover: (2)

- R is not a left introduction of B, because B is atomic.

- R is not a right introduction of A, because A is atomic.
- R is not a right introduction of $\neg\neg(A \to B)$, because the restriction (RC) on the context would not be satisfied.

Thus the only possible rule for R is a *pcut*. As the only non atomic formula at the place $*$ is $A \to B$, R is a *pcut* on $A \to B$ and p has the following form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B^{m'_1}|(A \to B)^{n_1}, A, A^{p_1}|\neg\neg(A \to B)^{q_1}, A, A^{r_1} \quad \dots\end{array}}{B^{m'}, A \to B|A \to B, (A \to B)^{n_0}, A^{p_0}|\neg\neg(A \to B), \neg\neg(A \to B)^{q_0}, A^{r_0}}\,Int^- \qquad \dfrac{\begin{array}{c}B|B|B\\ \vdots\, W\\ B, B^{m''}, A|A, B|B\end{array}}{B, B^{m''}|A \to B|A \to B}\,Int^+}{B, B^{m_0}|A \to B, (A \to B)^{n_0}, A^{p_0}|\neg\neg(A \to B), \neg\neg(A \to B)^{q_0}, A^{r_0}}\,pcut$$

Indeed:

- By the restriction (RC) on the contexts of the premises of Int^+, the occurrences of A and $A \to B$ at the place $*$ and the occurrences of A and $\neg\neg(A \to B$ at the place 1 can not be anywhere else than in the conclusion of Int^-.
- B must have at least one occurrence in the conclusion of Int^+ because the sequent $|A \to B|A \to B$ is not provable (if we give the value 1 to A and 0 to B the sequent is not satisfied).

Thus the premise of Int^+ is obtained by weakenings from the axiom $B|B|B$.

On the leftmost branch of p, we then go on with another sequent of the form (1), and the previous reasoning may be done again. Passing through all *pcuts* on the leftmost branch of the proof, we get a list $\Pi, \Pi_1, \dots, \Pi_l$ o (f sequents of the form (1).
Let us consider now the first rule applied above Π_l which is neither a weakening nor a contraction: by (2) it must be a *pcut*, which is impossible. ∎

7 Conclusion

We presented an intuitionistic sequent calculus for finitely many-valued logics. To get a calculus with positive semantics and which is sound and complete for a natural extension of Kripke models, we had to introduce a new deduction rule (the *pseudo-cut*) which can not exist in the two-valued case. Nevertheless the restriction (RC) on the right introductions may be seen as a generalization of the restriction which defines a two-valued intuitionistic calculus from a classical calculus.
The next step in this work is to investigate the computational meaning (if there is any) of this calculus. In particular, what do pseudo-cuts mean from a computational point of view ?

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