A New Spectrum of Recursive Models

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Abstract

We describe a strongly minimal theory S in an effective language such that, in the chain of countable models of S, only the second model has a computable presentation. Thus there is a spectrum of an ω_1–categorical theory which is neither upward nor downward closed. We also give an upper bound on the complexity of spectra.

Keywords: Strongly minimal, ω_1–categorical, spectrum of recursive models

1 Introduction

Our main purpose is to find a strongly minimal theory in an effective language whose spectrum of recursive models is the set $\{1\}$. We rely on some concepts in Khoussainov, Nies, and Shore [2], reviewed here briefly. Baldwin and Lachlan [1] showed that the countable models of an ω_1–categorical theory T form an $\omega + 1$–chain $M_0(T) \prec M_1(T) \prec \ldots \prec M_\omega(T)$ under elementary embeddings. In [2], we defined the spectrum of computable models of T,

$$\mathrm{SRM}(T) = \{i \leq \omega : M_i(T) \text{ has a computable presentation}\}.$$

We gave an example of an ω_1–categorical (in fact, strongly minimal) theory T such that $\mathrm{SRM}(T) = (\omega - \{0\}) \cup \{\omega\}$. Kudeiberganov [3], extending a result of Goncharov, proved that, for each $n \in \omega$, $n \geq 1$, there is an ω_1–categorical theory T such that $\mathrm{SRM}(T) = \{0, \ldots, n-1\}$. Here, in a priority construction, we combine the techniques used to prove the two results and obtain a strongly minimal theory T such that $\mathrm{SRM}(T) = \{1\}$. Thus, only $M_1(T)$ has a computable presentation (which we build in the priority construction).
The ultimate goal of these investigations is to describe all possible spectra of ω_1–categorical theories. In a sense, our example is the most complicated one found so far, since all the previous spectra were upward closed or downward closed in $\omega + 1$. Before we proceed to the main result, we give an upper bound on the complexity of spectra. Many ω_1–categorical theories are model complete (for instance, ACF_0, or, more generally, each ω_1–categorical theory axiomatizable by Π_2–formulas, by Lindström's test), so we also give a tighter upper bound for such theories.

Proposition 1.1
Suppose T is ω_1–categorical theory in an effective language. Then
(i) $\mathrm{SRM}(T) \in \Sigma^0_3(\emptyset^\omega)$
(ii) If T is model complete, then $\mathrm{SRM}(T) \in \Sigma^0_4$.

PROOF. Suppose $\beta(x)$ is a strongly minimal formula for T in the sense of [1]. Choose an effective numbering of the set D of atomic relations and negations of atomic relations in the given effective language over the domain $\mathbb{N}$ (typical elements of D are $fn = fgm$ and $\neg Rnm$, where $n, m \in \mathbb{N}$, f, g are unary function symbols and R is a binary relation symbol in our language). If we view a c.e. set W as a subset of D, then W gives rise to a presentation of a computable model, provided that exactly one of an atomic relation or its negation is in W, and, if the language contains an equality symbol $\approx$, then $\{n, m : n \approx m \in W\}$ is an equivalence relation compatible with W. The numbers e such that W_e determines a presentation form a Π^0_2–set P. For $e \in P$, this computable presentation is denoted by $\mathcal{A}_e$.
In the following, "a.i." stands for "algebraically independent" and, for a structure $\mathcal{A}$ in our language, $\beta(\mathcal{A})$ denotes $\{a \in \mathcal{A} : \mathcal{A} \models \beta(a)\}$. Let S_k be the group of permutations of $\{1, \ldots, k\}$.
To prove (i), we can suppose that $T \leq_T \emptyset^\omega$, otherwise $\mathrm{SRM}(T) = \emptyset$. Now $n \in \mathrm{SRM}(T) \Leftrightarrow \exists e \in P$

$$\mathcal{A}_e \models T\ [\Pi^0_1(\emptyset^\omega)]\ \& \tag{1.1}$$

$$\exists a_1, \ldots, a_n \in \beta(\mathcal{A}_e)[(a_1, \ldots, a_n) \text{ a.i.}]\ \& \tag{1.2}$$

$$\neg\exists a_1, \ldots, a_{n+1} \in \beta(\mathcal{A}_e)[a_1, \ldots, a_{n+1} \text{ a.i.}] \tag{1.3}$$

Also, in $\mathcal{A}_e$, $c_1, \ldots, c_k$ are a.i. iff for all formulas $\forall\varphi(x_1, \ldots, x_k)$

$$\forall\pi \in S_k[\mathcal{A}_e \models \varphi(c_{\pi(1)}, \ldots, c_{\pi(k)}) \Rightarrow \exists^\infty c\ \mathcal{A}_e \models \varphi(c_{\pi(1)}, \ldots, c_{\pi(k-1)}, c)],$$

which is $\Pi^0_1(\emptyset^\omega)$. Therefore (1.2) is $\Sigma^0_2(\emptyset^\omega)$, (1.3) is $\Pi^0_2(\emptyset^\omega)$ and the whole expression is $\Sigma^0_3(\emptyset^\omega)$, as desired.
For (ii), if T is model complete, then each formula α is equivalent under T to a Σ_1–formula σ with the same free variables. Moreover, by a recursive procedure using $T \cap \Pi_2$ as an oracle (the set of Π_2–sentences in T), we can determine such a σ from α. If T has a recursive model, then $T \cap \Sigma_1$ is c.e. and $T \cap \Pi_2$ is Π^0_2. Thus $T \leq_T \emptyset''$. Now, in the expression above, $\mathcal{A}_e \models T$ becomes Π^0_3. Moreover, since we can assume that all formulas involved are Σ_1, "$c_1, \ldots, c_k$ a. i." becomes Π^0_2, (1.2) becomes Σ^0_3 and (1.3) Π^0_3. ∎

2 $\{1\}$ is a spectrum

THEOREM 2.1
There is a strongly minimal (and hence ω_1–categorical) theory T in an effective language such that $M_i(T)$ $(i \leq \omega)$ has a computable presentation iff $i = 1$.

PROOF. We use a language consisting of binary relations P_k $(k \geq 0)$ called *edge relations* and further relations L_e $(e \geq 0)$. T contains axioms saying that the relations do not depend on the order of the elements, and only can hold for distinct elements. Let $\mathcal{L}_P$ be first-order language over $\{P_k : k \geq 0\}$. The models of T restricted to $\mathcal{L}_P$ are, with a small notational change, as in [2]. They consist of a disjoint union of components C_i, D. C_0 is a singleton, and C_{n+1} is the union of two copies of C_n, where elements in different subcomponents are connected via a P_n–edge. We call the models C_i *complexes of dimension* i, or for short, *i-complexes*, which replace the

i–cubes in [2] to simplify notation. There are natural embeddings of an i–complex into an $i+1$–complex. The ∞–complex D is the union of a chain of an i–complex for each finite i.

We determine T by describing a recursive presentation of $M_1(T)$. However, as in [2], $T \cap \mathcal{L}_P$ can be axiomatized by saying for which $n \in \omega$ an n–complex exists, and that there is at most one for each n. As in [2], for an infinite set $S \subseteq \omega$, let $\mathcal{A}_S = \bigcup_{n \in S} C_n$ be the $\mathcal{L}_P$–structure consisting of exactly one n–complex whenever $n \in S$. Then $T \cap \mathcal{L}_p = \mathrm{Th}(\mathcal{A}_S)$ is ω_1–categorical, where $M_i(T \cap \mathcal{L}_p)$ $(i \leq \omega)$ consists of $\mathcal{A}_S$ and an ∞–complex for each $j < i$.

An axiomatization for the theory in the full language is obtained by specifying, in addition, first–order definitions for the relations L_e. This is needed to show that the full theory is ω_1–categorical. Actually, $T \cap \mathcal{L}_P$, and hence T, are strongly minimal, as the following proposition shows. ∎

Proposition 2.2
For each infinite $S \subseteq \omega$, $\mathrm{Th}(\mathcal{A}_S)$ is strongly minimal.

Proof. Suppose M is a countable model of $\mathrm{Th}(\mathcal{A}_S)$ and $D \subseteq M$ is definable from parameters $a_0, \ldots, a_{n-1} \in M$ using edge relations among $P_0, \ldots, P_k$, $k \in S$, with the intent of showing that D is finite or cofinite. Now $\widetilde{M} = M \restriction \{P_0, \ldots, P_k\}$ consists of at most k complexes of dimension $< k$ and infinitely many k–complexes. Let F be the union of the complexes of dimension $< k$ and all complexes containing some a_i. Then F is finite, and if $c, d \in M - F$, there is an automorphism of $\widetilde{M}$ taking c to d and fixing each parameter used to define D. Thus, if $D \not\subseteq F$, then $D \cup F = M$. ∎

We now describe the construction of a computable presentation for $M_1(T)$. The ∞-complex will be the complex containing 0 in this particular representation.

The construction is in stages. Each stage s has finitely many substages, denoted by the letters τ, σ, which are numbered 0,1,2, ... through the whole construction, independently of s. $M_{1,\tau}(T)$ is the model obtained by the end of stage τ and has as a domain an initial segment $[0, u) \subseteq \mathbb{N}$, $u \geq \tau$. At the end of any substage τ, D will denote the current complex containing 0. If x is already in the domain, $\dim_\tau(x)$ denotes the dimension of the complex x is in at stage τ (so that $\dim_\tau(0)$ is the current dimension of D). The distance of x, y in the domain of $M_{1,\tau}(T)$ is

$$\begin{aligned} d_\tau(x,y) &= 0 \text{ if } x = y \\ d_\tau(x,y) &= k \text{ if } P_{k-1}xy \ (k \text{ is unique}) \\ d_\tau(x,y) &= \infty \text{ if there is no such } k. \end{aligned}$$

A complex C_r which exists at substage τ will be isomorphic to the "ball" $\{x \in D : d_\tau(0,x) \leq r\}$.

During the construction, we may do one of the following: (a) add a new m–complex (whose domain consists of the least numbers not used before), or (b) merge an existing complex C_r into D, using a procedure *Merge*(C_r), which chooses k large, first expands C_r, D to complexes D', D'' of dimension $k-1$ and then connects all elements of D' with all elements of D'' via P_{k-1}. Thus, $\dim_\tau(x)$ can change at most once from a constant value to "unbounded", while $d_\tau(x,y)$ may change once from ∞ to a finite value. We denote the limit value of $\dim_\tau(x)$ by $\dim(x)$ and the limit value of $d_\tau(x,y)$ by $d(x,y)$.

We recall a further definition from [2].

DEFINITION 2.3
A function f is **limitwise monotonic** if there exists a recursive function $\varphi(x,t)$ such that $\varphi(x,t) \leq \varphi(x,t+1)$ for all $x,t \in \omega$, $\lim_t \varphi(x,t)$ exists for every $x \in \omega$ and $f(x) = \lim_t \varphi(x,t)$.

Let S be the set of dimensions of finite complexes in any model of T. In [2, Lemma 2.2] we show that, if the prime model $\mathcal{A}_S$ is recursive, then the set S is the range of a limitwise monotonic function.
Let $\varphi_e(x,t)$, $e \in \omega$, be a uniform enumeration of all partial recursive functions φ such that for all $t' \geq t$ if $\varphi(x,t')$ is defined, then $\varphi(x,t)$ is defined and $\varphi(x,t) \leq \varphi(x,t')$. To ensure $M_0(T) \restriction \mathcal{L}_P$ (and hence $M_0(T)$) has no computable presentation, we satisfy requirements N_i which imply that S is not the range of a limitwise monotonic function given by φ_i.

$$N_i : \exists x,t \ \varphi_i(x,t) \text{ undefined } \vee \ \exists x \ \lim_t \varphi_i(x,t) \notin S.$$

The last disjunct may be achieved by ensuring $\lim_t \varphi_i(x,t) = \infty$.
An N_i-strategy has a parameter $m = m(N_i)$, whose values are chosen in a decreasing way in the interval $[g(i), g(i+1))$, where $g(i) = \sum_{j<i} h(j)$ and $h(j)$ is a computable function bounding the possible number of injuries to the requirement N_j (see Lemma 2.5 below). It has also parameters x,t. All parameters may be undefined.
The N_i–strategy is as in the proof of the recursion theoretic lemma [2, Lemma 2.1], but here it is incorporated into the priority construction of a presentation for $M_1(T)$. First add an m-complex, for an approriate m. The "opponent" now has to provide x,t such that $\varphi_i(x,t) = m$. As a response, use x to drive the limit $\lim_{t'} \varphi_i(x,t')$ to infinity. To do so, remove an m'-complex whenever $\varphi_i(x,t') = m'$ for $t' > t$. (The m'–complex was created by N_i itself, in which case $m' = m$, or by a lower priority N-strategy still waiting for the opponent's first move, which is now injured.) In some more detail, the N_i–strategy is the following. If any of the cases below applies, take the corresponding action.

(N1) All parameters are undefined, and $g(i+1) \leq s$.
Action. Let m be the largest unused number in $[g(i), g(i+1))$. Perform the procedure *Expand*$(\emptyset, m)$, which creates a new complex of dimension m.

(N2) m is defined, but x,t are undefined, and now $\varphi_{i,s}(x,t) = m$ for some $x,t < s$.
Action. Choose x,t as values for the parameters. Call the procedure
Merge(C_m), which puts C_m into D and thereby removes m from the list of possible values for $\lim_{t'} \varphi_i(x,t')$.

(N3) x,t are defined and now $\varphi_i(x,t') = m'$ for some $t' > t$, where currently an m'-complex $\neq D$ exists.
Action. Perform *Merge*$(C_{m'})$.

The requirements R_e code K into any presentation of a model $M_i(T)$, $i \geq 2$. By meeting the following requirements, we ensure that, if $e \notin K$, L_e is empty in each model of T, and if $e \in K$ then $L_e uv$ holds for any two algebraically independent elements of a model of T.
$R_e : \ e \in K \ \Rightarrow \ \exists n$

$$\forall x,y[dim(x) < g(e+1) \vee dim(y) < g(e+1) \vee d(x,y) < n \vee L_e xy]. \tag{2.1}$$

Since (2.1) can be expressed in a first-order way, and only the last alternative can occur for algebraically independent u, v, meeting all the requirements R_e is sufficient for the coding of K.

The R_e– strategy has a single parameter n_e, which is defined first at a stage s when $e \in K_s$ and may be made undefined finitely often by higher priority N–type strategies. The limit value will provide the witness n for (2.1). The R_e–strategy tries to ensure $L_e uv$ whenever n_e is defined, $\dim(u), \dim(v) \geq g(e)$ and $d(u,v) \geq n_e$. The priority ordering of the requirements is $N_0 \prec R_0 \prec N_1 \prec R_1 \prec \ldots$ Both types of requirements are *reset* by making all their parameters undefined.

Suppose an N-strategy wants to merge a complex $C_{m'}$ created by an N'–strategy into D, so that $N \prec N'$. This conflicts with the R_e–strategy in case we did not declare $L_e xy$ for all $x \in C_{m'}, y \in D$, since we will use an edge relation P_u, $u \geq n_e$ to connect x, y. It is too late now to add $L_e xy$, since we want a computable presentation of $M_1(T)$. This conflict is solved as follows: for all the requirements N' such that $R_e \prec N'$ and $C_{m(N')}$ exists, when e is enumerated into K, R_e first merges $C_{m(N')}$ into D. Only then R_e defines the first value of n_e, larger than all indices of edge relations used so far. If $N' \prec R_e$, before merging $C_{m(N')}$ into D we make n_e undefined, and R_e redefines it with large value after the merging took place.

The effect of R_e on the theory is described by a set F_e, which is cofinite if $e \in K$ and empty otherwise. Let $F_{e,0} = \emptyset$ and $F_{e,\tau} =$

$$F_{e,\tau-1} \cup \{k : n_e \text{ defined at substage } \tau \ \& \ k \geq n_e \ \& \ P_{k-1} \text{ first used at } \tau\}. \qquad (2.2)$$

Let $F_e = \bigcup_\tau F_{e,\tau}$. We will verify that

$$M_1 \models L_e xy \Leftrightarrow \dim(x), \dim(y) \geq g(e+1) \ \& \ d(x,y) \in F_e \cup \{\infty\}, \qquad (2.3)$$

which gives the desired first–order definition of L_e from finitely many edge relations. During a substage τ of the construction, we add $L_e xy$ to the presentation if y is a new element, $e \in K_{s-1}$ and (2.3) holds at that stage. Then presentation is computable. We need to verify that $L_e^{M_1(T)}$ actually satisfies (2.3), despite possible changes of $\dim(x), \dim(y)$ and $d(x,y)$ after τ.

We describe the procedures and the construction in detail. Whenever a procedure adds new numbers to the domain, they are chosen minimal in $\mathbb{N}$.

Expand(C_u, k) has as an input a u-complex C_u (recall that C_u is isomorphic to $\{x : d(x,0) \leq u\}$). It expands C_u to a k-complex by adding new elements and the appropriate P_{k-1}–relations between elements. We also include as a special case *Expand*$(\emptyset, k)$, which creates a new complex of dimension k. This is counted as one substage.

Merge(C_r) assumes that there is a complex C_r, $r = m(N)$ for some (unique) N. It merges C_r and D, but in a way that the overall goal that L_e be definable by (2.3) can be achieved. Let $N' \succ N$ be the requirement of highest priority such that $m(N')$ is defined. Recursively, call *Merge*$(C_{m(N')})$, using finitely many substages (if N' fails to exist, this step is vacuous). Next, in a single substage τ, perform the following:

1. Reset all the R-type requirements $\succ N$.
2. Choose k large and call *Expand*$(C_{r,\tau-1}, k-1)$, producing a complex D', and call *Expand*$(D_{\tau-1}, k-1)$ producing a complex D''.

3. Connect D', D'' by adding symmetric edges $P_{k-1}xy$, whenever x is in one and y in the other. This yields D_τ. Reset the requirement N.

The construction.
Stage 0. Let $D = \{0\}$, $\tau = 0$.
Stage $s > 0$. In the following, increase τ by one after each substage. Declare L_exy whenever an element y is added at a substage τ to the domain M_1 such that $e \in K_{s-1}$, $d_\tau(x,y) \in F_{e,\tau} \cup \{\infty\}$ and $\dim_\tau(x), \dim_\tau(y) \geq g(e+1)$.

a) Pick the requirement of highest priority U (if there is any) for which one of the following applies, and carry out the corresponding action,
   1. U is N_i, all parameters of N_i are undefined, and $g(i+1) \leq s$. *Action.* (N1) above.
   2. U is R_e, n_e was not defined up to now and $e \in K_s$. *Action.* Let $N \succ R_e$ be the N-type requirement of highest priority such that $m(N)$ is defined. If N exists, perform *Merge*$(C_{m(N)})$. Next, pick a large number $\geq g(e+1)$ as n_e.

b) If some u-complex exists and some N-type requirement desires to merge it via (N2) or (N3), perform *Merge*(C_u) for the minimal such u.

c) To ensure that $dim(D) \geq s$ at the end of stage s, call
   Expand$(D_{\tau-1}, \dim_\tau(0)+1)$

d) For all R_e such that n_e is now undefined but was defined before, redefine n_e with a large value.

The verification. We write M_i instead of $M_i(T)$.

Lemma 2.4
The model M_1 is recursive.

Proof. We want to test whether $M_1 \models Rxy$, where $x, y \in \mathbb{N}$ and R is a relation symbol from our language. We can suppose that $x < y$ and $y \in \mathrm{dom}(M_{1,\tau}) - \mathrm{dom}(M_{1,\tau-1})$ so that y is added at a substage τ of a stage s.
1. If R is P_k, then we distinguish two cases. If y is added by a procedure *Expand*, then $M_1 \models P_kxy \Leftrightarrow M_{1,\tau} \models P_kxy$. Otherwise $M_1 \models P_kxy$ because we connected D', D'' in a *Merge* procedure and $x \in D', y \in D''$, or we performed (c) of the construction during a stage $t \geq s$, in which case $k \geq s$ (since at each stage we introduce a new edge relation). Thus it suffices to check whether $M_{1,\max(k+1,s)} \models P_kxy$.
2. Now suppose R is L_e. Then, by the construction, $M_1 \models L_exy \Leftrightarrow M_{1,\tau} \models L_exy$ (since we determine whether L_exy holds when y is introduced). ∎

Lemma 2.5
There is a computable function h such that N_i is reset at most $h(i)$ times. In particular, during the construction, there is always a sufficient supply of candidates for the parameter $m(N_i)$, and also R_i is reset only finitely many times.

Proof. Let $h(0) = 0$. To determine $h(i+1)$, we observe that N_{i+1} can be reset at most two times before N_i is reset as well. For, if N_i is not reset, then either N_{i+1} was reset by R_i, which can only happen once, or during a *Merge* procedure for the sake of N_i. This means that before the merging, N_i has parameters x, t and $\varphi_i(x,t) = m(N_{i+1})$. By the way $m(N_{i+1})$ is chosen and since $\varphi_i(x,t)$ is nondecreasing in t, this can only happen once before $x(N_i)$ is changed.

Now, defining h recursively by $h(0) = 0$, $h(i+1) = 3h(i) + 3$, we obtain the desired bound. ∎

LEMMA 2.6
The requirements N_i are met. Hence $M_0(T)$ has no computable presentation.

PROOF. Suppose that N_i is not reset from stage s_0 on. Then N_i has permanently highest priority from s_0 on, and can therefore always fulfill its desire to create a complex. Since a complex $C_{m(N_j)}$, $N_i \prec N_j$ is merged into D whenever N_i desires, N_i is met. ∎

LEMMA 2.7
L_e is definable in all models of T by a Σ_1-formula in the restricted language $\mathcal{L}_P$, which depends only on e.

PROOF. Since $T = \mathrm{Th}(M_1)$ by definition, it suffices to work in M_1. Clearly, if $e \notin K$, then $L_e = \emptyset$. Now suppose $e \in K$. Then F_e is cofinite by Lemma 2.5. As discussed after (2.3), we want to prove that, for each $x, y \in \mathbb{N}$,

$$M_1 \models L_e xy \;\Leftrightarrow\; \dim(x), \dim(y) \geq g(e+1) \;\&\; d(x,y) \in F_e \cup \{\infty\}. \tag{2.4}$$

This suffices, for $\dim(x) \geq g(e+1)$ can be expressed by a Σ_1–formula in $\mathcal{L}_P$, and, if $\mathbb{N} - F_e \subseteq \{0, \ldots, m-1\}$, $m > 1$, then $d(x,y) \in F_e \cup \{\infty\} \;\Leftrightarrow\; d(x,y) \geq m \Leftrightarrow \neg P_0 xy \;\&\; \ldots \;\&\; \neg P_{m-2} xy$.
As in Lemma 2.4, suppose that $x < y$ and $y \in \mathrm{dom}(M_{1,\tau}) - \mathrm{dom}(M_{1,\tau-1})$ (but note that possibly $\dim_\tau(y) < \dim_\tau(x)$). We denote by $\mathrm{Compl}_\sigma(z)$ the complex z is in by the end of substage σ.
For the direction from left to right, if $L_e xy$, then at the end of substage τ, the right hand side in (2.4) holds. Thus $\dim_\sigma(x), \dim_\sigma(y) \geq g(e+1)$ for all $\sigma \geq \tau$, since dimension is nondecreasing over substages. Moreover, if $d_\tau(x,y) \in F_{e,\tau}$ then $d(x,y) = d_\tau(x,y) \in F_e$. Suppose now that $d_\tau(x,y) = \infty$ (so that x, y are in different complexes at the end of τ), but $k = d(x,y)$ is finite. Then at some substage $\sigma > \tau$, $\mathrm{Compl}_\tau(x) = \mathrm{Compl}_{\sigma-1}(x) = C_{m(N)}$ is merged into D during a run of the *Merge* procedure, while $y \in D_{\sigma-1}$. (If, instead, y enters D while x is in D already, we argue similarly.) During σ, x is in a complex D' which is connected with $D'' \supseteq D_{\sigma-1}$ using P_{k-1}, where k is chosen large. Note that $R_e \prec N$, since $\dim_\sigma(x) \geq g(e+1)$. So, during the run of the *Merge* procedure, n_e is still defined at σ when we use P_{k-1} and $k \geq n_e$, hence $k \in F_e$.
Now suppose the right hand side in (2.4) holds. We show $L_e xy$.
1. If $\dim_\tau(x) \leq \dim_\tau(y)$ and $\dim_\tau(x) < g(e+1)$, then, at the end of substage τ, the numbers x, y are in different complexes, otherwise $d(x,y) = d_\tau(x,y) < g(e+1)$ while $\min(F_e) \geq g(e+1)$. Suppose at the end of substage τ, $x \in C_{m(N)}$, and $y \in D$ or $y \in C_{m(N')}$, where $N \prec N'$. Since $\dim(x) \geq g(e+1)$, at a stage $\sigma > \tau$ we merge $C_{m(N)} = \mathrm{Compl}_\tau(x) = \mathrm{Compl}_{\sigma-1}(x)$ into D while $y \in D_{\sigma-1}$. By the *Merge* procedure and since $N \prec R_e$, R_e was reset before we merged $\mathrm{Compl}_\tau(x)$. So we add a relation $P_{k-1}xy$ while n_e is undefined, whence $k = d(x,y) \notin F_e$, contradiction. If $\dim_\tau(y) \leq \dim_\tau(x)$ and $\dim_\tau(y) < g(e+1)$, we argue similarly. We can henceforth assume that $\dim_\tau(x), \dim_\tau(y) \geq g(e+1)$, so that, by the end of stage τ, x is in D or in some $C_{m(N)}$ for some $N \succ R_e$, and similarly for y.

2. If $d_\tau(x,y) = \infty$ and we do not declare $L_e xy$ at τ, then n_e is undefined at τ. If $e \in K_{s-1}$ then, by the end of stage $s-1$, n_e was defined, and we made n_e undefined at substages of s prior to τ. Then, while performing *Merge*, we would have merged the complexes x and y are in into D at a substage of s prior to σ, contrary to $d_\tau(x,y) = \infty$. If $e \notin K_{s-1}$, then, since $e \in K$, by (a.2) in the construction, we merge the distinct complexes $\mathrm{Compl}_\tau(x)$ and $\mathrm{Compl}_\tau(y)$ into D at some substage before we define n_e for the first time. So, again $d(x,y) \notin F_e$.
3. Finally suppose $d_\tau(x,y) = k < \infty$. Since $k \in F_e$ and P_{k-1} is used first at a substage $\leq \tau$, $k \in F_{e,\tau}$. Since y is added at τ, we declare $L_e xy$. ∎

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