A Model Theoretic Approach to Translations Between Logics

MARCELO ESTEBAN CONIGLIO, *Group of Theoretical and Applied Logic, Department of Philosophy, State University of Campinas, P.O. Box 6133, 13081-970, Campinas, SP, Brazil.*
E-mail: coniglio@cle.unicamp.br

WALTER ALEXANDRE CARNIELLI, *Group of Theoretical and Applied Logic, Department of Philosophy and Center for Logic, Epistemology and the History of Science, State University of Campinas, P.O. Box 6133, 13081-970, Campinas, SP, Brazil.*
E-mail: carniell@cle.unicamp.br

Abstract

We use the language and concepts of model theory to describe the relationship between logics through translations. Abstract logics are conceived as two-sorted first-order structures, and translations between logics as morphisms of structures. This leads to a formal framework for studying abstract logics and translations between them, appropriate to give precise content to the slogan: *if* T *is a translation between logics* L_1 *and* L_2*, then the meaning of* L_1 *is preserved in* L_2.

Keywords: abstract logics, combinations of logics, translations between logics, model theory

1 Basic motivation

The concept of translation between logics (introduced in [7], see also [5] for comments) is a very broad one, and several properties of abstract logics, such as relative consistency and the meaning of a logical system in terms of others (via *possible-translations semantics*, cf. [4]), can be stated in terms of translations. We are using here the notion of *abstract logic* in the sense of the seminal paper [3] by Brown and Suszko.

However, there is no theory capable of expressing the general properties of abstract logics and explaining the full meaning of translations between them. In this paper we discuss this problem, and offer a formal framework where logics and their translations can be treated and their properties expressed. This section recalls some basic definitions and results from (many-sorted) model theory. We assume the reader is familiar with the notion of a *two-sorted first-order structure* as a 5-uple

$$\mathrm{A} = \langle A_1, A_2, \mathcal{P}, \mathcal{F}, \mathcal{C}\rangle.$$

The (first-order, two-sorted) language $\mathbb{L}$ associated with a two-sorted structure and the notion of satisfiability in a two-sorted structure are defined as usual. The sorts of $\mathbb{L}$ will be denoted by $\mathbf{A_1}$ and $\mathbf{A_2}$. We also assume familiarity with related notions such as: *restrictions* $\mathrm{A}_{|\langle B_1, B_2\rangle}$ for $\emptyset \neq B_i \subseteq A_i$ $(i = 1, 2)$, and *expansions*.

Definition 1.1
Let $\mathrm{A} = \langle A_1, A_2, \mathcal{P}, \mathcal{F}, \mathcal{C}\rangle$ and $\mathrm{B} = \langle B_1, B_2, \mathcal{P}', \mathcal{F}', \mathcal{C}'\rangle$ be two-sorted structures for the same language $\mathbb{L}$. A *morphism of structures* $h : \mathrm{A} \longrightarrow \mathrm{B}$ is a pair $h = \langle F_1, F_2\rangle$ such that:

(i) $F_i : A_i \longrightarrow B_i$ for $i = 1, 2$;
(ii) if P is a $(n+m)$-ary predicate symbol of $\mathbb{L}$, then

$$(\vec{\mathrm{a}}; \vec{\mathrm{b}}) \in P^{\mathrm{A}} \quad \text{implies that} \quad (F_1(a_1), ..., F_1(a_n); F_2(b_1), ..., F_2(b_m)) \in P^{\mathrm{B}}$$

for all $\vec{\mathrm{a}} = (a_1, ..., a_n) \in A_1^n$, $\vec{\mathrm{b}} = (b_1, ..., b_m) \in A_2^m$;
(iii) if c (of sort $\mathbf{A_1}$) and d (of sort $\mathbf{A_2}$) are symbols for constants, then

$$F_1(c^{\mathrm{A}}) = c^{\mathrm{B}} \quad \text{and} \quad F_2(d^{\mathrm{A}}) = d^{\mathrm{B}};$$

(iv) if $f : \mathbf{A_1}^n \times \mathbf{A_2}^m \longrightarrow \mathbf{A_1}$ is a symbol of function of $\mathbb{L}$, then

$$F_1(f^{\mathrm{A}}(\vec{\mathrm{a}}; \vec{\mathrm{b}})) = f^{\mathrm{B}}(F_1(a_1), ..., F_1(a_n); F_2(b_1), ..., F_2(b_m))$$

for all $\vec{\mathrm{a}} = (a_1, ..., a_n) \in A_1^n$ and $\vec{\mathrm{b}} = (b_1, ..., b_m) \in A_2^m$. An analogous condition is required if f has codomain of sort $\mathbf{A_2}$.

A morphism $h = \langle F_1, F_2\rangle$ is an *isomorphism* if F_1 and F_2 are both bijective, and "implies that" is substituted by "iff" in condition (ii) above. We say that A is *embeddable* in B, denoted by $\mathrm{A} \subseteq \mathrm{B}$, if there exists a morphism $h : \mathrm{A} \longrightarrow \mathrm{B}$ such that h is an isomorphism between A and $\mathrm{B}_{|\langle F_1(A_1), F_2(A_2)\rangle}$.

Proposition 1.2
If $\langle F_1, F_2\rangle : \mathrm{A} \longrightarrow \mathrm{B}$ is an isomorphism, then

$$\mathrm{A} \models \varphi[a_1, ..., a_n; b_1, ..., b_m] \quad \text{iff} \quad \mathrm{B} \models \varphi[F_1(a_1), ..., F_1(a_n); F_2(b_1), ..., F_2(b_m)]$$

for all formulas $\varphi(x_1, ..., x_n; y_1, ..., y_m)$ with at most $x_1, ..., x_n$ and $y_1, ..., y_m$ as free variables (of sort $\mathbf{A_1}$ and $\mathbf{A_2}$, respectively), for all $(a_1, ..., a_n) \in A_1^n$ and for all $(b_1, ..., b_m)$ in A_2^m. In particular, $\mathrm{A} \models \sigma$ iff $\mathrm{B} \models \sigma$ for all $\mathbb{L}$-sentences σ.

2 Logics as structures

In this section we show how model-theoretic concepts can express deep properties of abstract logics and translations between them.

Definition 2.1
An *abstract logic* L is a two-sorted structure of the form

$$\mathrm{L} = \langle X, Z, \in, \vdash, \cup, \{\cdot\}, \emptyset\rangle$$

where $Z \subseteq \mathcal{P}(X) = \{\Gamma \,:\, \Gamma \subseteq X\}$, $\in\ \subseteq X \times Z$ is the *belonging* relation, $\vdash\ \subseteq Z \times X$ is the *consequence* relation, $\cup : Z \times Z \longrightarrow Z$ is the *join* operation and $\{\cdot\} : X \longrightarrow Z$, $\{\cdot\}(x) = \{x\}$ for all $x \in X$. The relation $\in$ and the functions $\cup$ and $\{\cdot\}$ have their usual set-theoretic meaning, and $\emptyset$ is the empty set. Additionally, Z must be a Boolean algebra satisfying the following:

(i) $\emptyset \in Z$ and $X \in Z$;
(ii) $\{a\} \in Z$ for all $a \in X$;
(iii) if $\Gamma, \Delta \in Z$ then $\Gamma \cup \Delta \in Z$;
(iv) if $\Gamma \in Z$, then $X - \Gamma \in Z$;
(v) if $\Gamma \in Z$, then $\{a \in X \ : \ \Gamma \vdash a\} \in Z$.

Throughout the rest of this paper, $\mathbb{L}$ will denote the language

$$\{\mathbf{A_1}, \mathbf{A_2}, \in, \vdash, \cup, \{\cdot\}, \emptyset\}$$

appropriate to "talk about" abstract logics. Note that $\mathbb{L}$ contains the constant symbol $\emptyset$ (of sort $\mathbf{A_2}$) for the empty set. A structure of the type of abstract logics will be denoted by

$$\mathrm{A} = \langle X, Z, \in_{\mathrm{A}}, \vdash_{\mathrm{A}}, \cup_{\mathrm{A}}, \{\cdot\}_{\mathrm{A}}, \emptyset_{\mathrm{A}} \rangle,$$

omitting the subscripts in the set-theoretic operations if A is an abstract logic.

We will see that translations between logics are special cases of morphisms between structures.

Definition 2.2
Let $\mathrm{L}_i = \langle X_i, Z_i, \in, \vdash_i, \cup, \{\cdot\}, \emptyset\rangle$ be abstract logics ($i = 1, 2$). A *translation from* L_1 *to* L_2 is a morphism $h : \mathrm{L}_1 \longrightarrow \mathrm{L}_2$, $h = \langle T, IT\rangle$ such that

$$IT(\Gamma) = \{T(a) \ : \ a \in \Gamma\} \ (= T(\Gamma)) \qquad \text{for all } \Gamma \in Z_1.$$

An isomorphic translation between L_1 and L_2 is called a *strongly conservative translation from* L_1 *to* L_2. A translation h is *conservative* if h is a morphism satisfying "iff" instead of "implies that" in condition (ii) of 1.1.

It is worth observing that, if $h : \mathrm{L}_1 \longrightarrow \mathrm{L}_2$ is a translation, then

$$\Gamma \vdash_1 a \quad \text{implies} \quad T(\Gamma) \vdash_2 T(a),$$

and, if h is conservative, the following holds:

$$\Gamma \vdash_1 a \quad \text{iff} \quad T(\Gamma) \vdash_2 T(a),$$

that is, we recapture the usual definitions of translation and conservative translation introduced in [7]. It is also to be noted that the parameter IT is irrelevant in our definition of translation. On the other hand, the image of a logic under a conservative translation $T : \mathrm{L}_1 \longrightarrow \mathrm{L}_2$ is also a logic (T being conservative ensures that $T(X_1)$ satisfies condition 2.1(v)). T is a strongly conservative translation iff T is a bijective conservative translation. This kind of translation is very important, because it determines that the logics involved are the same (from the point of view of the language $\mathbb{L}$). In other words, the logics related by translations of this kind are indiscernible by sentences in the language $\mathbb{L}$ of logics.

PROPOSITION 2.3
Let $T : L_1 \longrightarrow L_2$ be a strongly conservative translation. Then for any formula $\varphi(x_1, ..., x_n; Y_1, ..., Y_m)$, every $(a_1, ..., a_n) \in X_1^n$ and $(\Gamma_1, ..., \Gamma_m)$ in Z_1^m,

$$L_1 \models \varphi[a_1, ..., a_n; \Gamma_1, ..., \Gamma_m] \quad \text{iff} \quad L_2 \models \varphi[T(a_1), ..., T(a_n); T(\Gamma_1), ..., T(\Gamma_m)],$$

where φ has at most $x_1, ..., x_n$ (of sort $\mathbf{A_1}$) and $Y_1, ..., Y_m$ (of sort $\mathbf{A_2}$) as free variables. In particular, for each sentence σ in the language $\mathbb{L}$,

$$L_1 \models \sigma \quad \text{iff} \quad L_2 \models \sigma.$$

PROOF. An immediate consequence of proposition 1.2. ∎

The definition of abstract logic does not impose any conditions on the consequence relation $\vdash$. If we set $C(\Gamma) = \{a \in X \ : \ \Gamma \vdash a\}$ then $C : Z \longrightarrow Z$ is the closure operator associated with L. Tarski's well-known axioms for a closure operator $C : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ are the following (cf. [7]):

[C1] $A \subseteq C(A)$ for all $A \subseteq X$;
[C2] $A \subseteq B$ implies $C(A) \subseteq C(B)$ for all $A, B \subseteq X$;
[C3] $C(C(A)) \subseteq C(A)$ for all $A \subseteq X$.

In the language $\mathbb{L}$ this can be expressed as follows:

[A1] $(\forall Y)(\forall x)(x \in Y \to Y \vdash x)$;
[A2] $(\forall Y_1)(\forall Y_2)(Y_1 \subseteq Y_2 \to (\forall x)(Y_1 \vdash x \ \to \ Y_2 \vdash x))$;
[A3] $(\forall Y_1)(\forall x)((\exists Y_2)[(\forall y)(y \in Y_2 \ \leftrightarrow \ Y_1 \vdash y) \ \wedge \ Y_2 \vdash x] \ \to \ Y_1 \vdash x)$,

where $Y_1 \subseteq Y_2$ represents the formula $(\forall x)(x \in Y_1 \to x \in Y_2)$. As a consequence of Proposition 2.3, if T is a strongly conservative translation from L_1 to L_2, then L_1 satisfies [A1]-[A3] iff L_2 satisfies [A1]-[A3]. That is, L_1 is a logic (in the sense of [7]) iff L_2 is a logic in that sense (provided that $Z_i = \mathcal{P}(X_i)$, $i = 1, 2$).
In general, properties of logics will be expressed by means of sentences in $\mathbb{L}$, that is, theories $\mathbb{T}$ over $\mathbb{L}$. For example, the fact that Z contains every closed theory can be expressed as

$$(\forall Y_1)(\exists Y_2)(\forall x)[(x \in Y_2) \leftrightarrow (Y_1 \vdash x)].$$

As another example, a theory $\mathbb{T}$ may contain the sentence

$$(\forall Y_1)(\forall Y_2)(\forall x)(\forall y)[(Y_1 \vdash x) \ \wedge \ (Y_2 \cup \{x\} \vdash y) \ \to \ (Y_1 \cup Y_2 \vdash y)]$$

expressing the usual cut rule, which we call **CUT** property. Therefore, L_1 has **CUT** iff L_2 has **CUT**, provided that there exists a strongly conservative translation between them.

3 Ultraproducts of Logics

The main result, to be proved in this section, is that the ultraproduct of abstract logics is an abstract logic. We assume that the reader is familiar with the usual concept of filter, ultrafilter and ultraproduct.

THEOREM 3.1
Let $\mathrm{F} = \{\mathrm{A}_i \;:\; i \in I\}$ be a family of two-sorted structures over the same language $\mathbb{L}_1$, and let A be the ultraproduct of F over an ultrafilter $D \subseteq \mathcal{P}(I)$. Then

$$\mathrm{A} \models \sigma \quad \text{iff} \quad \{i \in I \;:\; \mathrm{A}_i \models \sigma\} \in D$$

for every $\mathbb{L}_1$-sentence σ.

If we conceive the elements of D as "large" subsets of I, the Fundamental Theorem of Ultraproducts 3.1 says that the ultraproduct A satisfies a sentence σ iff σ is satisfied by a "large" number of structures A_i.

We now prove the important fact that the class of abstract logics is closed under ultraproducts.

THEOREM 3.2 (Ultraproducts of Abstract Logics)
Let $\mathrm{F} = \{\mathrm{L}_i \;:\; i \in I\}$ be a family of abstract logics, where

$$\mathrm{L}_i = \langle X_i, Z_i, \in, \vdash_i, \cup, \{\cdot\}, \emptyset\rangle \qquad (i \in I).$$

Then the ultraproduct L of F over an ultrafilter $D \subseteq \mathcal{P}(I)$ is an abstract logic.

PROOF. The ultraproduct L' of F over D is given by

$$\mathrm{L}' = \langle X_D, Y_D, \in_D, \vdash_D, \cup_D, \{\cdot\}_D, \emptyset_D\rangle,$$

where $X_D = \Pi_D\{X_i \;:\; i \in I\}$, $Y_D = \Pi_D\{Z_i \;:\; i \in I\}$,

$f_D \in_D \Gamma_D \quad \text{iff} \quad \{i \in I \;:\; f_i \in \Gamma_i\} \in D,$

$\Gamma_D \vdash_D f_D \quad \text{iff} \quad \{i \in I \;:\; \Gamma_i \vdash_i f_i\} \in D,$

$\Gamma_D \cup_D \Delta_D = \langle \Gamma_i \cup \Delta_i \;:\; i \in I\rangle_D = \{\Theta \;:\; \{i \in I \;:\; \Theta_i = \Gamma_i \cup \Delta_i\} \in D\},$

$\{f_D\}_D = \langle \{f_i\} \;:\; i \in I\rangle_D = \{\Gamma \;:\; \{i \in I \;:\; \Gamma_i = \{f_i\}\} \in D\},$

$\emptyset_D = \{\Gamma \;:\; \{i \in I \;:\; \Gamma_i = \emptyset\} \in D\}.$

Consider the 5-uple $\mathrm{L} = \langle X_D, Z_D, \in, \vdash^D, \cup, \{\cdot\}, \emptyset\rangle$ defined as follows:

$Z_D \subseteq \mathcal{P}(X_D)$, where $S \in Z_D$ iff there exists $\Gamma \in \prod_{i \in I} Z_i$ such that

$S = \{f_D \in X_D \;:\; \{i \in I \;:\; f_i \in \Gamma_i\} \in D\} = S(\Gamma);$

if $S = S(\Gamma)$, then $S \vdash^D f_D$ iff $\{i \in I \;:\; \Gamma_i \vdash_i f_i\} \in D$.

We assume that $\emptyset$, $\in$, $\cup$ and $\{\cdot\}$ are set-theoretic. We must prove that Z_D and $\vdash^D$ are well defined. Our argument is based on the following facts, whose proof we omit.

Fact a: If $f_D = g_D$ and $\Gamma \in \prod_{i \in I} Z_i$, then

$$\{i \in I \;:\; f_i \in \Gamma_i\} \in D \quad \text{iff} \quad \{i \in I \;:\; g_i \in \Gamma_i\} \in D.$$

Fact b: If $S(\Gamma) = S = S(\Delta)$, that is, if

$$\{f_D \in X_D \ : \ \{i \in I \ : \ f_i \in \Gamma_i\} \in D\} = \{f_D \in X_D \ : \ \{i \in I \ : \ f_i \in \Delta_i\} \in D\},$$

then $\Gamma_D = \Delta_D$ in Y_D.

Let $S = S(\Gamma) = S(\Delta) \in Z_D$ and $f_D \in X_D$. Then $\Gamma_D = \Delta_D$, by **Fact b**. Therefore

$\{i \in I \ : \ \Gamma_i \vdash_i f_i\} \in D$ iff $\Gamma_D \vdash_D f_D$
iff $\Delta_D \vdash_D f_D$ iff $\{i \in I \ : \ \Delta_i \vdash_i f_i\} \in D$.

This shows that $\vdash^D$ is a well defined relation over $Z_D \times X_D$, thus L is a two-sorted structure. We will prove that L is an abstract logic isomorphic to L$'$.

Let's define a map $S : Y_D \longrightarrow Z_D$, given by

$$S(\Gamma_D) = \{f_D \in X_D \ : \ \{i \in I \ : \ f_i \in \Gamma_i\} \in D\}.$$

The map S is well defined: if $\Gamma_D = \Delta_D$, then

$$R = \{i \in I \ : \ \Gamma_i = \Delta_i\} \in D.$$

On the other hand, suppose that $P = \{i \in I \ : \ f_i \in \Gamma_i\} \in D$. Then $R \cap P \in D$, that is,

$$\{i \in : \ f_i \in \Gamma_i \ \wedge \ \Gamma_i = \Delta_i\} \in D.$$

But clearly $R \cap P \subseteq \{i \in I \ : \ f_i \in \Delta_i\} = T$, thus $T \in D$. Analogously, $T \in D$ implies that $P \in D$. This proves that S is well defined. Moreover, since $S(\Gamma_D) = S(\Gamma)$ then S is injective, by **Fact b**, and S is clearly surjective. Thus, S is a bijective map. Consider $h = \langle id, S\rangle$, where $id : X_D \longrightarrow X_D$ is the identity map. We now prove that h is an isomorphism between L$'$ and L. In fact:

$f_D \in_D \Gamma_D$ iff $\{i \in I \ : \ f_i \in \Gamma_i\} \in D$ iff $id(f_D) \in S(\Gamma_D)$;
$\Gamma_D \vdash_D f_D$ iff $\{i \in I \ : \ \Gamma_i \vdash_i f_i\} \in D$ iff $S(\Gamma_D) \vdash^D id(f_D)$.

On the other hand,

$$S(\Gamma_D \cup_D \Delta_D) = \{f_D \ : \ \{i \in I \ : \ f_i \in \Gamma_i \cup \Delta_i\} \in D\}.$$

Let $f_D \in S(\Gamma_D \cup_D \Delta_D)$; then $R = \{i \in I \ : \ f_i \in \Gamma_i \cup \Delta_i\} \in D$. Let

$$P = \{i \in I \ : \ f_i \notin \Gamma_i\} \quad \text{and} \quad Q = \{i \in I \ : \ f_i \notin \Delta_i\}.$$

If $P \in D$ and $Q \in D$ then $P \cap Q \in D$, that is, $\{i \in I \ : \ f_i \notin \Gamma_i \cup \Delta_i\} \in D$, a contradiction (because $R \in D$). Thus $P \notin D$ or $Q \notin D$, whence

$$\{i \in I \ : \ f_i \in \Gamma_i\} \in D \quad \text{or} \quad \{i \in I \ : \ f_i \in \Delta_i\} \in D,$$

because D is an ultrafilter. This means that $f_D \in S(\Gamma_D) \cup S(\Delta_D)$. Conversely, if $f_D \in S(\Gamma_D)$, then $R' = \{i \in I \ : \ f_i \in \Gamma_i\} \in D$. But $R' \subseteq R$, thus $R \in D$, whence

$f_D \in S(\Gamma_D \cup_D \Delta_D)$; analogously, $f_D \in S(\Delta_D)$ implies that $f_D \in S(\Gamma_D \cup_D \Delta_D)$. That is, $S(\Gamma_D \cup_D \Delta_D) = S(\Gamma_D) \cup S(\Delta_D)$. Finally, observe that

$$\begin{aligned} S(\{f_D\}_D) &= \{g_D \ : \ \{i \in I \ : \ g_i \in \{f_i\}\} \in D\} \\ &= \{g_D \ : \ \{i \in I \ : \ g_i = f_i\} \in D\} = \{f_D\} = \{id(f_D)\}. \end{aligned}$$

Since $S(\emptyset_D) = \emptyset$, we infer that h is an isomorphism. Moreover, the proof above shows that Z_D satisfies:

$\emptyset \in Z_D$;
$\{f_D\} \in Z_D$ for every $f_D \in X_D$;
$\mathcal{S}, \mathcal{R} \in Z_D$ implies that $\mathcal{S} \cup \mathcal{R} \in Z_D$.

Since $X_i \in Z_i$ for all $i \in I$ then

$$\{i \in I \ : \ \mathrm{L}_i \models (\exists Y)(\forall x)(x \in Y \leftrightarrow (x = x))\} = I \in D,$$

therefore $\mathrm{L} \models (\exists Y)(\forall x)(x \in Y \leftrightarrow (x = x))$, by Theorem 3.1 and Proposition 1.2. That is, $X_D \in Z_D$. Since each Z_i satisfies item (iv) of Definition 2.1 then

$$\{i \in I \ : \ \mathrm{L}_i \models (\forall Y_1)(\exists Y_2)(\forall x)(x \in Y_2 \leftrightarrow x \notin Y_1)\} = I \in D,$$

therefore $\mathrm{L} \models (\forall Y_1)(\exists Y_2)(\forall x)(x \in Y_2 \leftrightarrow x \notin Y_1)$, by Theorem 3.1 and Proposition 1.2. That is, Z_D satisfies (iv). Finally, since each Z_i satisfies 2.1(v), then

$$\{i \in I \ : \ \mathrm{L}_i \models (\forall Y_1)(\exists Y_2)(\forall x)[(x \in Y_2) \leftrightarrow (Y_1 \vdash x)]\} = I \in D,$$

therefore $\mathrm{L} \models (\forall Y_1)(\exists Y_2)(\forall x)[(x \in Y_2) \leftrightarrow (Y_1 \vdash x)]$, by Theorem 3.1 and Proposition 1.2. That is, Z_D satisfies (v). Thus L is an abstract logic, and this concludes our proof. ∎

COROLLARY 3.3
If L is the ultraproduct of the family $\{\mathrm{L}_i\}_{i \in I}$ over D, then

$$\mathrm{L} \models \sigma \quad \text{iff} \quad \{i \in I \ : \ \mathrm{L}_i \models \sigma\} \in D$$

for every $\mathbb{L}$-sentence σ.

PROOF. Immediate from Theorem 3.1. ∎

This means that the ultraproduct logic L satisfies a property σ (expressed in the language $\mathbb{L}$) iff the property σ is verified by a subset of logics L_i that is "large" (in some sense).

4 Some additional results from model theory

In this section we present some additional results from model theory which can be applied to our general approach to logics as structures. All new results (which appear as corollaries) are immediate, and the proofs are omitted.

Definition 4.1
Let φ be a formula of a two-sorted language $\mathbb{L}'$.
(a) φ is a Π_1^0-*formula* if $\varphi = (\forall x_1 \cdots \forall x_n)\psi$, where each x_i is a variable (of sort $\mathbf{A_1}$ or sort $\mathbf{A_2}$) and ψ has no quantifiers.
(b) φ is a Π_2^0-*formula* if $\varphi = (\forall x_1 \cdots \forall x_n)(\exists y_1 \cdots \exists y_m)\psi$, where each x_i and y_j is a variable (of sort $\mathbf{A_1}$ or sort $\mathbf{A_2}$) and ψ has no quantifiers.
(c) φ is a *positive formula* if φ has no $\rightarrow$ and $\neg$ symbols.

Proposition 4.2
Let $\mathbb{T}$ be a closed theory. If $\mathbb{T}$ is axiomatizable by Π_1^0-formulas, then $\mathbb{T}$ is preserved by submodels.

Corollary 4.3
Let L be a logic. If L satisfies a set $\mathbb{T}$ of Π_1^0-formulas, then every sublogic L$'$ of L also satisfies $\mathbb{T}$.

Definition 4.4
A *chain of structures* is a sequence of $\mathbb{L}'$-structures $\mathcal{S} = \{\mathrm{A}_\gamma : \gamma < \alpha\}$ such that A_γ is a submodel of A_β (if $\gamma < \beta < \alpha$). The *union* A *of* $\mathcal{S}$ is the structure given by $A = \bigcup_{\gamma<\alpha} A_\gamma$, $B = \bigcup_{\gamma<\alpha} B_\gamma$, and the relations and functions are the union of the respective relations and functions of the structures in $\mathcal{S}$. The constants are unchanged.

Proposition 4.5
Any set of Π_2^0-formulas is preserved by union of chains. $\square$

Corollary 4.6
The union of a chain of logics is a logic. $\square$

Corollary 4.7
If $\mathcal{S}$ is a chain of logics, each of them satisfying a set $\mathbb{T}$ of Π_2^0-formulas, then the union of $\mathcal{S}$ also satisfies $\mathbb{T}$. $\square$

Proposition 4.8
Let $\mathbb{T}$ be a consistent closed theory. Then, $\mathbb{T}$ is preserved by homomorphisms (that is: $\mathrm{A} \models \mathbb{T}$ implies that $\mathrm{B}_{|\langle F_1(A_1), F_2(A_2)\rangle} \models \mathbb{T}$ for every $\langle F_1, F_2\rangle : \mathrm{A} \longrightarrow \mathrm{B}$) iff $\mathbb{T}$ has a positive set of axioms. $\square$

The following corollary establishes an important property of translations: the contents of positive sentences are preserved under translations.

Corollary 4.9
Given a translation $T : \mathrm{L} \longrightarrow \mathrm{L}'$, then

$$\mathrm{L} \models \varphi[a_1, ..., a_n; \Gamma_1, ..., \Gamma_m]$$

implies that

$$\mathrm{L}' \models \varphi[T(a_1), ..., T(a_n); T(\Gamma_1), ..., T(\Gamma_m)]$$

for every positive formula φ, every $(a_1, ..., a_n) \in X^n$ and every $(\Gamma_1, ..., \Gamma_m) \in Z^m$. Therefore, if a logic L satisfies a set $\mathbb{T}$ of positive properties, then, for every conservative translation $T : \mathrm{L} \longrightarrow \mathrm{L}'$, the image of L by T also satisfies $\mathbb{T}$.

5 Examples: first-order properties of abstract logics

As a direct application we show how general properties of abstract logics (or universal logics, cf. [2]) can be treated by our approach. From now on the expression $Y_1 \vdash Y_2$ will denote the formula $(\forall x)(x \in Y_2 \rightarrow Y_1 \vdash x)$.

DEFINITION 5.1
Following [2] we define the following properties of abstract logics:

$[A]$ $(\forall x)(\{x\} \vdash x)$ (Auto-deductivity);

$[M\infty]$ $(\forall Y_1)(\forall Y_2)(\forall x)[((Y_1 \vdash x) \wedge (Y_1 \subseteq Y_2)) \rightarrow Y_2 \vdash x]$
(Infinite monotonicity);

$[Y\infty]$ $(\forall Y_1)(\forall Y_2)(\forall x)[((Y_1 \vdash Y_2) \wedge (Y_2 \vdash x)) \rightarrow Y_1 \vdash x]$
(Infinite syllogism);

$[AMY] \equiv_{def} [A] \wedge [M\infty] \wedge [Y\infty]$;

$[A\infty]$ $(\forall Y)(\forall x)(x \in Y \rightarrow Y \vdash x)$
(Infinite auto-deductivity);

$[H\infty]$ $(\forall Y_1)(\forall Y_2)(\forall Y_3)(\forall x)[((Y_1 \vdash Y_2) \wedge (Y_2 \cup Y_3 \vdash x)) \rightarrow Y_1 \cup Y_3 \vdash x]$
(Infinite Herz's law);

$[N]$ $(\forall Y_1)(\forall x)[(Y_1 \vdash x) \leftrightarrow (\forall Y_2)((Y_2 \vdash Y_1) \rightarrow (Y_2 \vdash x))]$
(Normal law).

PROPOSITION 5.2
(i) $[A\infty]$ and $[Y\infty]$ implies $[M\infty]$.
(ii) $[A\infty]$ implies that $[Y\infty]$ is equivalent to $[H\infty]$.
(iii) $[N]$ is equivalent to $[AMY]$.

PROOF. By adapting the proofs of these results found in [2] to our first-order formalism. ∎

6 Fibring propositional calculi

Finally, we will show that the process of fibring propositional calculi introduced by Sernadas et al. (cf. [8]) can also be expressed in our framework.

DEFINITION 6.1
Let $C = \{C_k\}_{k \in \mathbb{N}}$ be a (propositional-based) signature (cf. [8]). That is, each C_k is a set of connectives of arity k. Connectives in C_0 are *propositional symbols*. Let $\mathbb{L}_C$ be the language obtained from $\mathbb{L}$ and C, considering each $f \in C_k$ as a symbol of function $f : \mathbf{A_1}^k \longrightarrow \mathbf{A_1}$. In particular, each $p \in C_0$ is a symbol of constant of sort $\mathbf{A_1}$. A *Hilbert calculus specification* is a pair $\langle C, R \rangle$ where C is a (propositional-based) signature and R is a set of sentences in the language $\mathbb{L}_C$ of the form

$$(\forall)(\{\xi_1, ..., \xi_n\} \vdash \xi) \quad \text{and} \quad (\forall)(\emptyset \vdash \xi),$$

where $(\forall)\Phi$ denotes the universal closure of the formula Φ, and $\{\xi_1, ..., \xi_n\}$ denotes the term $(\cdots(\{\xi_1\} \cup \cdots) \cup \{\xi_n\})$. Given a specification $E = \langle C, R\rangle$ we obtain the logic $\mathrm{L}_E = \langle X, Z, \in, \vdash, \cup, \{\cdot\}, \emptyset, C\rangle$ over $\mathbb{L}_C$ defined as follows:

- X is the free algebra generated by C;
- $Z = \mathcal{P}(X)$;
- $\vdash = \bigcap\{\vdash' \subseteq Z \times X \mid \langle X, Z, \in, \vdash', \cup, \{\cdot\}, \emptyset, C\rangle \models R \cup \{[A\infty], [Y\infty]\}\}$.

Observe that, by proposition 5.2, $\mathrm{L}_E \models [M\infty]$. Given $\Gamma \cup \{\alpha\} \in Z$ we say that α *is provable from* Γ *in* L_E if $\mathrm{L}_E \models (Y_1 \vdash x_1)[\Gamma, \alpha]$.

The *unconstrained fibring* of two logics L_1 and L_2 over $\langle C^1, R_1\rangle$ and $\langle C^2, R_2\rangle$ is defined naturally as $\mathrm{L} = \mathrm{L}_1 \vee \mathrm{L}_2$, the logic obtained from the specification $\langle C, R\rangle$, where $C = C^1 \sqcup C^2$ is the (disjoint) union of C^1 and C^2 and $R = R_1 \cup R_2$. It is easy to see that $\mathrm{L}_1 \vee \mathrm{L}_2$ is the least logic L over $\mathbb{L}_C$ satisfying $\mathrm{L}_i \subseteq \mathrm{L}_{|\mathbb{L}_{C^i}}$ for $i = 1, 2$, that is: if $\mathrm{L}_i \subseteq \mathrm{L}'_{|\mathbb{L}_{C^i}}$ $(i = 1, 2)$ then $\mathrm{L} \subseteq \mathrm{L}'$.

The *constrained fibring* of L_1 and L_2, obtained by sharing connectives (cf. [8]), is obtained analogously. The generalisation to arbitrary families of logics is straightforward.

7 Concluding remarks

We have introduced a formal framework in which abstract logics and their translations can be analysed and (hopefully) better understood. Our definitions and main results show that:

(1) most properties of abstract logics can be expressed and proved in our formalism;

(2) classical results of model theory can be used to establish important (meta)properties of translations between abstract logics, helping to assign a philosophical meaning to the concept of translation;

(3) interesting results from the point of view of combinations of abstract logics can also be obtained. In particular, we have shown that the category of abstract logics is closed under ultraproducts and that the process of fibring is also expressable in the proposed formalism.

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