Computability, Algorithms and Complexity

Department of Computing, Imperial College, London SW7 2BZ, UK.

Contents

Books and other reading iv
Introduction 1
1. What is an algorithm? 3
1.1. The Problem 3
1.2. What is an algorithm? 6
1.3. Turing machines 8
1.4. Why use Turing machines? 12
1.5. Church's thesis 13
1.6. Summary of Section 17
2. Turing machines and examples 17
2.1. What exactly is a Turing machine? 18
2.2. Input and Output of a Turing machine 18
2.3. Representing Turing machines 21
2.4. Examples of Turing machines 24
2.5. Summary of Section 35
3. Variants of Turing machines 36
3.1. Computational power 36
3.2. Two-way 37
3.3. Multi-tape 41
3.4. Other variants 51
3.5. Summary of Section 53
4. Universal Turing machines 54
4.1. Standard Turing machines 54
4.2. Codes for standard Turing machines 55
4.3. The universal Turing machine 58
4.4. Coding 60
4.5. Summary of Section 62
5. Unsolvable problems 63
5.1. Introduction 63
5.2. The halting problem 64
5.3. Reduction 69
5.4. Gödel's Incompleteness Theorem 76
5.5. Summary of Section 82
Part I in a nutshell 82

Part II: Algorithms 85
6. Use of algorithms 85
6.1. Run time function of an algorithm 85
6.2. Choice of algorithm 91
6.3. Implementation 91
6.4. Summary of Section 93
7. Graph algorithms 93
7.1. Graphs: the basics 93
7.2. Representing graphs 95
7.3. Algorithm for searching a graph 95
7.4. Paths and Connectedness 102
7.5. Trees, Spanning trees 103
7.6. Complete graphs 106
7.7. Hamiltonian circuit problem (HCP) 107
7.8. Summary of Section 108
8. Weighted graphs 108
8.1. Example of Weighted Graph 108
8.2. Minimal spanning tree 109
8.4. Shortest path 117
8.5. Travelling salesman problem (TSP) 117
8.6. Polynomial time reduction 118
8.7. NP-completeness taster 120
8.8. Summary of Section 120
Part II in a nutshell 120

Part III: Complexity theory 122
Introduction 122
9 Basic complexity theory 123
9.1 Yes/no problems 124
9.2 Polynomial time Turing machines 126
9.3 Tractability of Problems 127
10 Non-deterministic Turing machines 133
10.1 Definition of non-deterministic TM 133
10.2 Examples of NDTMs 135
10.3 Speed of NDTMs 137
10.4 Simulation of NDTMs by ordinary TMs 139
11 Reduction in p-time 142
11.1 Definition of p-time reduction 142
11.2 Examples 143
11.3 ≤ is a pre-order 144
11.4 Closure of NP 146

11.5 The P-problems are ≤-easiest 148
11.6 Summary 150
12 NP-completeness 150
12.1 Introduction 150
12.2 Proving NP-completeness by Reduction 153
12.3 Cook's theorem 154
Sample exam questions 158
Part III in a nutshell 160

BOOKS AND OTHER READING

Texts

V.J. Rayward-Smith, *A first course in computability*, Blackwell, paperback, 1986 (£12.99).
An introductory paperback that covers parts I & III of the course, and some of part II. More detailed than this course.

D. Harel, *The Science of Computing*, Addison-Wesley, 1989 (£16.95).
A good book for background and motivation, with fair coverage of this course and a great deal more. Less detailed than this course.

Advanced/reference texts

Robert Sedgewick, *Algorithms*, Addison-Wesley, 2nd ed., 1988.
A practical guide to many useful algorithms and their implementation. A reference for part II of the course.

J. Bell, M. Machover, *A course in mathematical logic*, North Holland, 1977.
A good mathematical text, for those who wish to read beyond the course.

G. Boolos, R. Jeffrey, *Computability and Logic*, Cambridge University Press, 1974.
A thorough text, but mathematically demanding.

M.R. Garey, D.S. Johnson, *Computers and intractibility — a guide to NP-completeness*, Freeman, 1979.
The 'NP-completeness bible'. For reference in Part III.

J.E. Hopcroft & J.D. Ullman, *Introduction to automata theory, languages and computation*, Addison-Wesley, 1979.
A classic text with a wealth of detail; but it concentrates on abstract languages and so has a different approach from ours.

Papers

A.M. Turing, *On computable numbers with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society (Series 2), vol. 42 (1936), pp. 230-265.
One of the founding papers of computer science, but very readable. Contains interesting philosophical reflections on the subject, the first description of the Turing machine, and a proof that some problems are unsolvable.

G. Boolos, Notices of American Mathematical Society 36 no. 4 (April 1989), pp 388-390. A new proof of Gödel's incompleteness theorem.

Popular material

A. Hodges, *Enigma* Vintage, 2nd edition, 1992. A recent biography of Alan Turing.
Readable and explains some key ideas from this course (e.g. the halting problem) in clear terms.

R. Penrose, *The Emperor's New Mind*, Vintage, ~£9. Mainly physics but describes Turing machines in enough rigour to cover most of Part I of this course (e.g., halting problem). Enjoyable, in any case.

INDEX

3SAT 153
Aer Lingus 94, 96
al-Khwārazmi 6, 13, 127
algorithm 3, 6, 13, 15, 17, 85
  formalising 6
  genetic 152
  other definitions of 15
  probabilistic 129, 152
array 30
average case 85, 91, 122, 152
Babbage 6
binary numbers 35
blank 9, 19
  in code of TM 56
breadth first 96, 101, 139
Breaking the Code 2
Church 5, 7, 16, 81
Church's thesis 8, 14, 15, 17, 63, 69, 74, 123, 126, 127
  disproving 15
  Evidence 15, 36, 38, 45, 53, 139
  formal statement 21
Church-Turing thesis 8
co-NP 153
code of configuration 78
code of TM 55
  ambiguity of 56
  checking 56
  Generality of 57
  non-standard TM 57
  other codings 58
  redundancy in 58
  uniqueness of 57
code(a) 61
code(S) 55, 64
code-breaking 54
coding of alphabets 61
complement of a yes-no problem 131
complement of a yes/no problem 128
complexity 13, 50, 63, 123
composite (i.e., not prime) 129, 130, 135, 138
computable function 14
computational power 37
connected component 103, 104
contradiction 5, 64, 66, 67, 69, 77, 78
Cook 123, 127, 151, 153, 154
Cook's theorem 78, 154
Cook-Karp thesis 126, 127
Cray 12, 37, 90
cycle 103, 105
decision problem 43
delta (d) 11
depth first 96, 99
Dijkstra 117
Directed graph 94, 95
edge (of graph) 94
EDIT 73
EIHP 73
empty word 19, 73
equivalence class 103
equivalence relation 89, 93, 103, 146
Exam questions 50, 68, 74, 158
Exhaustive search 131, 139
exponential time 86, 88, 91, 107, 117, 122, 127, 129, 130
Fermat's last theorem 68
Flowcharts of Turing machines 21
forest 104
fringe 96, 99, 100
Gödel number 58
Gödel's theorem 2, 5, 64, 76, 154
Goldbach's conjecture 68, 77, 125
graph 93
  complete 106, 109
  connected 102, 104, 105, 110
  disconnected 102
  weighted 109
gravity 15

halting problem 5, 63, 64, 69, 77
  empty-input 73
  is hard 67
  uniform 74
Hamiltonian circuit 107, 117
Hamiltonian Circuit Problem (HCP) 107, 123, 124, 129, 130, 138, 143, 147, 151, 153
Harel iv, 44, 117, 129
HCP 153
head 73
head (function) 19, 23
heap 97, 101
Hilbert 6
Hofstadter 2, 12
Hollerith 6
Hopcroft & Ullman iv, 37
instruction table 18
  partial function 18
interpreter 4
intractable 120, 122, 128, 131
irreflexive 93
Karp 128
Kleene 16
König's tree lemma 138
Kruskal 116
Ladner 151
lambda calculus 16, 53
linked list, 95
log linear time 86, 88, 91, 102, 131
logic 63, 154, 157
loop 1, 10, 11
maze 96, 106
mimic 45
mimic or simulate 37
M[w] 70
Newton 7, 15
node 93
Non-primality 129, 135, 151
NP 122, 132, 138, 139, 146, 147, 150, 154
NP-complete 5, 108, 120, 123, 130, 151, 153, 154
NPC 151
O-notation 89, 126
oracles 70
P 122, 128
  closed under complementation 128
p-time 122, 126, 127, 128, 129, 135, 137, 138, 139, 142, 146
P=NP 122, 139, 150, 151, 152, 153
palindrome 43
paradoxes 3, 64, 68
parameter (in state) 25, 47
  finitely many values 27
partial recursive functions 53
path 102
  length 102
  non-backtracking 102
  shortest 117
Penrose 2, 67
Pianola 6
polynomial 126, 145, 147
  degree 127
polynomial time 107, 117, 118
Post 16
  correspondence problem 81
pre-order 89, 144
Prim's algorithm 111, 115, 116
Primality problem 129, 151
priority queue 97, 115
  entry 97
  label 97, 98, 100
propositional satisfaction (PSAT) 123, 129, 130, 137, 138, 151, 152, 153, 154
pseudo-code 22
punched cards 6
queue 96, 97
Rayward-Smith iv, 37, 82, 157
reactive systems 17
reduction 64, 69, 74, 142

of HCP to TSP 119
p-time 118, 123, 142, 148, 152, 153, 154
Examples 143
for NPC 153
for P 149
in NP 146
reflexive 144
transitive 144
reflexive 93
register machines 36, 53
relation (binary) 93
equivalence 93
irreflexive 93
reflexive 93
symmetric 93
transitive 93
reverse (of a word) 35
satisfiable (formula) 130, 137, 154
scratch characters 19, 61
elimination of 61
Sedgewick 85, 92, 95
separation property 111, 113
Shifting 25, 31
simulate 45, 59, 62, 139
solvable 73
square 0
marking of 32
implicit 32
in standard TM 55
stack 96, 97
standard character set 54
string comparison 27, 31
symmetric 93
tail 19, 28, 70
Tarjan 95
theta notation 89
three-SAT (3SAT) 153
timeM(n) 126
tracks 29, 33
finitely many 30
in standard TM 55
removing 33
use in simulating 2-tape TM 46
tractable 127, 131
transitive 93
Travelling salesman problem (TSP) 117, 123, 124, 129, 130, 137, 138, 143, 148, 151
flat version 152
tree 103, 139
e=n-1 105
minimal spanning (MST) 110, 111, 113, 114, 131
spanning 104, 117
Turing iv, 2, 7, 16, 54
Turing machine 8, 36
accept 124, 135
alphabet 9
coding 55
configuration 11, 78, 140, 155
code of 78
current square 9
current state. 9
equivalence of two ~s 36, 74
formal definition of 18
function computed by 20
halting state 11
halts and fails 11, 20
halts and succeeds, 11
head 9
idealised computer 13
initial state 9
input 19, 134
input-output function of 20
instruction table 11, 133
hard-coding of 48
in code of TM 55
move 10
move left from square 0 11
Multi-tape 41

as bounded concurrent system 45
as concurrent system 41
simulation by 1-tape TM 46
no applicable instruction 11
non-deterministic 16, 53, 133
example 135
output 11, 20, 134
undefined 20
polynomial time 126
read 10
reject 124, 135
run 10, 126, 133, 139
of NDTM 135, 137, 138
run time function 126
running forever 11
solve (a y/n problem) 125
standard 54, 55, 58, 60, 61, 62, 64
in EIHP 73
in HP 64
state 9
Subroutines 34
tape 9
two-dimensional 30, 51
Two-way-infinite tape 37
universal 54, 58, 64, 125
write 10
Turing-computable 21
UHP 74
Unary numbers 24, 34, 35, 144
converting to binary 35
unsolvable problems 5, 63
vertex 93
visit algorithm 95, 104, 115
visited (node) 96
Warning 22, 23, 24, 30, 34, 42, 55, 65, 67, 70, 100, 107, 115, 118, 143
weight (of edge) 94, 95, 109
Weighted graph 94
word 18
worst case 85, 118, 126, 130
X–Y property 111
yes/no problem 43, 124, 134
complementary problem 128
no-instance 124
solve 125, 135
yes-instance 124