161, Winter, 2005–6
Homework 4, Due Feb 8

Exercise 1 (2.1 (Chapt. 3) (15 points)) Let $n \in \mathbb{N}$. Show that there is no $k \in \mathbb{N}$ such that $n < k < n+1$.

Exercise 2 (2.2 (Chapt. 3) (15 points)) Use the previous exercise to prove for all $m, n \in \mathbb{N}$: if $m < n$, then $m+1 \leq n$. Conclude that $m < n$ implies $m+1 < n+1$ and that therefore the successor $S(n) = n+1$ defines a one-to-one function on $\mathbb{N}$.

Exercise 3 (2.4 (Chapt. 3) (20 points)) Prove that for every $n \in \mathbb{N}$, $n \neq 0, 1$, there is a unique $k \in \mathbb{N}$ such that $n = (k+1)+1$.

Exercise 4 (3.1 (Chapt. 3) (25 points)) Let f be an infinite sequence of elements of A, where A is ordered by $\prec$. Assume that $f_n \prec f_{n+1}$ for all $n \in \mathbb{N}$. Prove that $n < m$ implies $f(n) \prec f(m)$ for all $n, m \in \mathbb{N}$. [*Hint:* Use induction on m in the form of Exercise 2.11 of the textbook, with $k = n+1$.]

Exercise 5 (3.2 (Chapt. 3) (25 points)) Let $(A, \prec)$ be a linearly ordered set and $p, q \in A$. We say that q is a *successor* of p if $p \prec q$ and there is no $r \in A$ sucht that $p \prec r \prec q$. Note that each $p \in A$ can have at most one successor. Assume that $(A, \prec)$ is nonempty and has the following properties:

(a) Every $p \in A$ has a successor.

(b) Every nonempty subset of A has a $\prec$-least element.

(c) If $p \in A$ is not the $\prec$-least element of A, then p is a successor of some $q \in A$.

Prove that $(A, \prec)$ is isomorphic to $(\mathbb{N}, <)$. Show that the conclusion need not hold if one of the conditions (a)–(c) is omitted.