MODELING AND RENDERING THE GROWTH OF SPELEOTHEMS IN REAL-TIME

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Abstract: Among the many natural phenomena already studied in computer graphics research, the interior of caves remains largely unexplored. We present in this paper a plausible approach, based on Geological studies, for modeling the genesis and growth of speleothems. Speleothems are mineral depositions formed in caves, such as stalactites, stalagmites and columns. Our approach takes advantage of the new geometry shader, present in the new generation of GPUs, and can generate in a controlled way, a large variety of cave scenes, with meaningful geological parameters.

1 INTRODUCTION

In Computer Graphics as a research field, researchers have always been attracted to reproduce the beauty of Nature and the intrinsic complexity in many several phenomena, such as rain simulation (Rousseau et al., 2006), illumination models and shading for clouds (Bouthors et al., 2006), fire simulation (Min and Metaxas, 2007), among many more phenomena. Since the initial efforts in this subject (Blinn et al., 1983), it was obvious that the simulation of natural phenomena is scientifically a great challenge, because it potentially deals with a great volume of data and a high computational demand, if all the mathematical and physical complexity of the real models are taken into consideration. Nature is able to create many complex details. One example of this complexity is the genesis and growth of speleothems such as stalactites and stalagmites, found in the interior of limestone caves. Among the many natural phenomena already studied in computer graphics research, the interior of caves remains largely unexplored. We present in this paper a plausible approach, based on Geological studies, for modeling the genesis and growth of speleothems. Speleothems are mineral depositions formed in caves, such as stalactites, stalagmites and columns. Our approach takes advantage of the new geometry shader, present in the new generation of GPUs, for growth of speleothems in realtime using meaningful geological parameters.

2 RELATED WORK

The great variety and beauty of speleothems structures has been a source of curiosity and a mystery for hundreds or even thousands of years. Scientific studies in such different areas as Physics, Chemistry, Mathematics, and Geology, try to explain the origin and growth of these speleothems. In (Short et al., 2005b), Short and colleagues describe the process of stalactite growth as a free-boundary problem, analyzing the dynamics of the responsible fluids for its formation. In (Short et al., 2005a), the same group presented the first free-boundary approach to the axisymmetric growth of stalactites. They derived a law of motion in which the local growth rate depends on the radius and inclination of the stalactite's surface. This law holds under a set of limiting assumptions valid for typical stalactite growth conditions. Numerical studies of this surface dynamics showed the existence of an attractor in the space of shapes, toward which stalactites will be drawn regardless of initial conditions. Another work by the same group (Short et al., 2006), describes the stalactite genesis and growth, with parameters similar to the ones described in (Short et al., 2005b), specific for ice stalactites, known as icicles. In this approach, the shape of the icicles also depends on the initial parameters and of their future interations.

In (Szilder and Lozowski, 1995), a discreet three-dimensional model was developed using a mathematical model of Random-Walk to simulate the growth of icicles. That model is based on a series of parameters supplied in the beginning of the simulation. The values of those parameters vary with time along the simulation, and they affect the shape of the stalactite model directly. The water flow along the icicle's surface is divided into fluid elements which follow a stochastic path towards the icicle tip. During its motion, a fluid element may freeze on the icicle's lateral surface or at its tip. The fluid elements may also drip from the icicle tip.

In a rare example of computer graphics research in this topic, a pioneering work by Yang and Ouhyoung used fractal theory to model stalactites and stalagmites (Ouhyoung and Yang, 1993; Yang and Ouhyoung, 1992). To model the peculiar irregular shape of speleothems, they used recursive and random subdivision on an initial simple geometric model. They also used radiosity techniques (Sillion and Puech, 1994) for illumination of the scene. They emphasize that stalagmites are more difficult to model because of their much higher complexity. Therefore instead of using fractal subdivision, they simulated these following the natural process of erosion and accumulation. In the results section we compare our results with one of these papers ((Yang and Ouhyoung, 1992)) since we did not have access to the other one. Finally, in a paper about an intuitive painting interface for making local deformations to 3D surfaces (Lawrence and Funkhouser, 2004), the authors illustrate one possible result with a cave environment with stalactites and stalagmites.

3 GEOLOGICAL BACKGROUND

According to (Lino, 2001), speleothems are mineral depositions formed in caves, basically by chemical processes of dissolution and precipitation. This characteristic gives them a character of great permanence, and which may even be structural. The most common speleothems are stalactites, stalagmites and columns.

3.1 How the speleothems grow

In order to penetrate into the fractures of the limestone rock, the rainwater, acidulated by atmospheric and soil carbon-dioxide (CO2), dissolves it and carries off the calcium carbonate until it finally emerges on the roof of the cave. The drop of water suspended on the roof of the cave is exposed to environmental conditions, such as greater ventilation, alterations in temperature, pH, and CO2 pressure. These environmental conditions create chemical instability through the liberation of the CO2 into the cave and the consequent precipitation of part of the dissolved carbonate. The drop of water hangs on the roof until it reaches the volume and weight necessary to overcome surface tension and fall. Hanging on the roof of the cave and exposed to environmental conditions of the interior's cave, the surface of the drop develops the first crystals of calcite; these, organizing themselves during the period in which the drop is still in contact with the roof, form an initial crystalline ring which will serve as a base for a future stalactite. Drop by drop, a hollow tubular stalactite grows in a downward direction. The drop, when it at last falls, carries with it a solution of carbonate which slowly forms a sucession of layers on the floor immediately below, and which becomes a stalagmite. The opposing growth of stalactites and stalagmites might finally result in the union of the two, to form a column. Figure 1(a) and (b) show examples of *stalactites* and *stalagmites*, respectively, whereas (c) illustrates a *column*, formed by the union of a stalactite with a stalagmite.

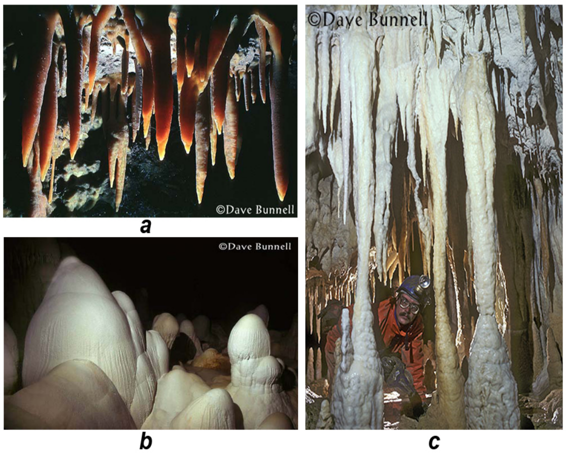

Figure 1: Real Speleothems: (a) Stalactites - (b) Stalagmites - (c) Columns.

3.2 Speleothems Growth Stages

Still according to (Lino, 2001), the process of formation of the speleothems is divided into three different stages (Figure 2). Each one of these stages have determinant variables that interfere somehow in the shape, dimensions, coloration and texture of the speleothems formed through the mechanism of calcite precipitation.

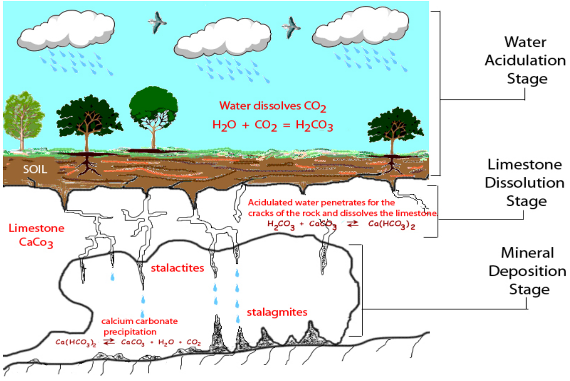

Figure 2: Speleothems growth stages.

Water Acidulation Stage: In the first stage, we have to consider the external environment located above of the interior of the limestone cave. The pluviometric index, rain periodicity, soil vegetal coverage, average temperature, and the chemistry composition of the soil, among other less important factors, will influence the generated spheleotems. A bigger volume and regularity of the rain, determines an increase in the flow rate of acidulated water that reachs the cave roof. The drops remain a short time suspended in the cave roof, just to be quickly replaced by others. Thus, a little quantity of carbonate will be deposited in the cave roof. In other words, considering a reasonable saturation level, the stalactites in this place will be small or even not present, and the stalagmites will have a fast growth and/or voluminous shape. On the other hand, if the water volume decreases, the tendency is that the stalactites are larger than the stalagmites. Also, if there aren't enough amount and regularity of rain, neither one of the speleothems will be formed or they will have their growth interrupted until satisfactory rain values are reached.

Limestone Dissolution Stage: In the second stage, we consider the thickness of the rocky layer, degree of purity, crack level, and solubility. This properties will conditionate the saturation level of the solution, the access channels to the cave, and the kind of minerals that will be dissolved and blended with the saturated solution, thus defining the characteristic of the speleothem that will be formed. The degree of purity of the rocky layer, for example, can influence directly the coloring of the speleothems. In general, speleothems have a white coloration due to their calcareous nature. However, the presence of others minerals such as iron, copper, manganese and malachite, when blended with the solution, add a characteristic pigmentation to the speleothem.

Mineral Deposition Stage: In the third stage, factors related to the interior of cave will define the typology of the speleothems. The first aspect to consider is the morphology and dimensions of the underground space - planar or inclined roof, abyss, great saloons, floor under water, etc. For example, to achieve the cave, if a drip of saturated solution does not find a propitious environment for which there is the dripping (interior space), the process of growing is interrupted. The second aspect to consider is the interrelation between the internal cave environment and the external cave environment. This interrelation implicates in larger or smaller water and air circulation. Thus, regularity and perpendicularity of the speleothems, such as stalactites, can be modified due to circulation of air coming from the external environment between the corridors of the cave. Such circulation causes a deformation in the normal growth form of the stalactite, changing its direction, not only vertically, but also horizontally.

A better understanding of the concepts of formation of caves and speleothems in general, can be found in (Lino, 2001; Kaufmann, 2003; Self and Hill, 2003).

4 OUR MODEL

This section details how our model was developed and implemented. Since it is a simple model of the growth of speleothems, we used approximations to the real physical-chemical processes previously described. The goal of this model was to make an approximation of the geometric shape and appearance of speleothems, using the capabilities of the new pipeline architecture of the Direct3D 10 (Blythe, 2006; Boyd, 2007). Primarily, a exploration of the new capabilities of two pipeline stages: Geometry-Shader (GS) and Stream-Output (SO). The GS stage processes entire primitives. Its input is a full primitive (three vertices for a triangle, two vertices for a line, or a single vertex for a point). In addition, each primitive can also include the vertex data for any edge-adjacent primitives. This could include at most an additional three vertices for a triangle or an additional two vertices for a line. The GS also supports limited geometry amplification and de-amplification. Given an input primitive, the Geometry Shader can discard the primitive, or emit one or more new primitives. The SO stage is designed for streaming primitive data from the pipeline to memory on its way to the rasterizer. Data can be streamed out and/or passed into the rasterizer. Data streamed out to memory can be recirculated back into the pipeline as input data or read-back from the CPU.

4.1 Shape of the Speleothem

When it starts, the application sends to the graphics pipeline a primitive similar to a thin hollow cylinder illustrated in Figure 3. Such primitive will be the basis for the creation of all speleothems in the simulation. This primitive approximates the real initial calcite crystalline ring.

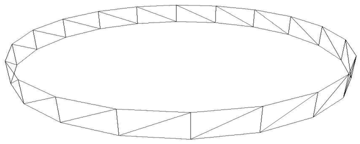

Figure 3: Base Primitive (cylinder).

Besides the basic properties of each vertex, such as position, normal vector, and texture coordinates, we included an ID for each vertex of the base primitive. These IDs will be a key element in the definition of the shapes of the speleothems, and are used to create the new primitives in the geometry-shader stage, as we explain below. Once inside the pipeline, the vertices pass through the vertex-shader stage and reach the geometry-shader stage. Once there, the vertices of the base will be used as information for creation of the new primitives. These primitives will form the body of the speleothem. First, all the IDs of the vertices of the primitive which are entering in this stage will be increased by two. Figure 4 shows, respectively, the vertices of the base primitive when entering the geometry-shader stage and, later, with their incremented IDs.

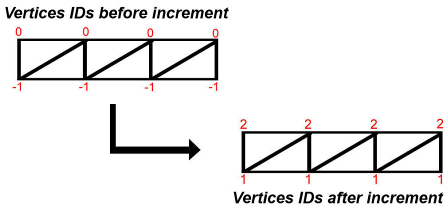

Figure 4: IDs increment.

After increasing the IDs, a new primitive is created based on the base primitive. This new cylinder is similar to the first one. However, its shape is controlled by two parameters generated by the application, *radius* and *height*. The radius parameter models the diameter of the new primitive (cylinder). This value is always minor or equal to the radius of the earlier primitive (for stalactites), and any value for stalagmites. This radius is randomly generated by the application, following the variations previously described.

The height parameter models the distance between the base primitive and a new generated primitive. The new primitive will be translated with respect to the negative $y - axis$ (stalactites) or to the positive $y - axis$ (stalagmites) according to the value of this parameter. The height, similarly to the radius, is randomly generated by the application and passed to the shader effect file as a constant buffer variable.

A summary of the possible effects on these variables (radius and height) and the type of structure generated is given in Table 1. The equation which defines the new primitive, based on the radius and height supplied by the application in relation to the earlier primitive is given by $v' = v + height + r * \vec{N}$, where v' is the new vertex position, v is the current vertex position, $\vec{N}$ is the normal vector, and r represents the radius of the new ring.

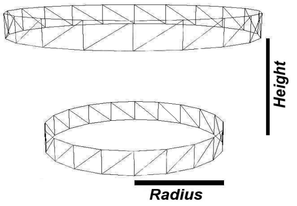

Figure 5: New generated primitive based in the parameters: Radius and Height.

Figure 5 shows the base primitive and a new generated primitive inside the geometry-shader stage. For the new cylinder, new IDs are assigned. The IDs for the new primitive will be exactly the same IDs of the base primitive when entering the pipeline for the first time, before the increment. In other words, highest vertices of the cylinder receive IDs = 0 (zero) and the lowest vertices receive IDs = -1. Finally, the last task in the geometry-shader stage is to connect the vertices of the first and second cylinders. We use the IDs to select only the vertices which will be processed for the creation of the new primitive in the ring sequence. Only vertices with IDs smaller than or equal to zero are processed. Figure 6 shows the two previously generated cylinders and the link of their vertices, making a cone-like shape. Also observe the configuration of the IDs after the first pass through the geometry-shader stage.

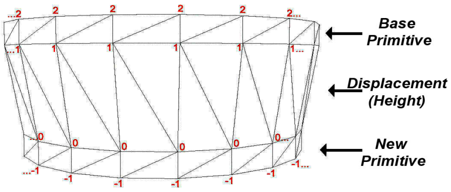

Figure 6: Cone-like shape created inside the geometry-shader stage.

Using an approximation of the real model of growth of speleothems described in (Kaufmann, 2003), the *Radius* and *Height* parameters, that control the speleothems growth rate and form, are defined as following. The pluviometric index level (PIL) of the water acidulation stage, described in Section 3.2, is the amount of rain water. To our model, the pluviometric index is one of three possible levels: **Level 0**: lack of rain, in other words, no growth of speleothems. **Level 1**: makes possible the creation of stalactites, but there is no dripping, in other words, the drop with saturated solution can solidify just in the tip or in the surface of the stalactite already formed. **Level 2**: Dripping of the saturated solution occurs due to the high volume of rain water, possible growth of stalagmites.

The drip interval depends on the pluviometric index, and the cracks in the limestone rock (fissures where the rain water flows downward to the cave environment). The drip interval is inversely proportional to the pluviometric index. If the drip interval is small (wet weather), the tendency is an increase of the new ring diameter. If the drip interval is long (dry weather), the tendency is a decrease of the new ring diameter. The *radius* parameter is linked with these properties, and it is responsible to create some deformities on the surface of the speleothem.

The calcium and carbon-dioxide concentration in the drop of the satured solution depend on the soil vegetal coverage, which is a source of carbon-dioxide (CO2) for rain water. A soil with greater vegetation cover increases the concentration of CO2 in water (greater acidulation), making minerals such as calcium and other minerals mixed with water that infiltrates in the layers of the limestone rock. The water is super-saturated of calcium and, on entering the cave, CO2 is degassed from the drop and the excess calcium is deposited as calcite (CaCO3) in the cave. The *height* parameter is linked with these properties. Thus, the height of the speleothems is proportional to the calcium and carbon-dioxide concentration.

Another possibility must be considered: if the calcium and carbon-dioxide concentration are low, the drops may solidify before reaching the tip of the stalactite or the stalagmite base. Thus, only horizontal growth takes place, rather than the vertical growth, exactly at the point of the body of the speleothem where the drip solidifies. This horizontal displacement is proportional to the value of radius of the new primitive, where the drip with the saturated solution solidified. This horizontal displacement creates some deformity on the surface of the speleothem. The equation of the horizontal displacement is given by: $vd = kf * arr/irr$ where vd is the vertex displacement factor, kf is a fixed scaling factor to adjust the size of displacements proportionally to the size of the speleothem, arr is the actual ring radius and irr is the initial ring radius. Stalactites growth is linked with two control variables: pluviometric index and CO2 concentration. Stalagmites growth is linked with both pluviometric index and CO2 concentration, and also with the drip interval. Table 1 shows all the properties and the values that affect the *height* and *radius* of the speleothems.

The presence of air circulation in the interior of the cave also affects the shape of the speleothem. The new primitive ring in formation can suffer a position displacement in the direction of the air flow. The parameter that models the presence or not of the air flow, as well as its direction, is supplied by the application or by the interactive intervention of the user. It is computed as $\vec{v}' = wi(\vec{v} + \vec{w})$ where $\vec{v}'$ is the new position of vertex $\vec{v}$, $\vec{w}$ is the unit vector defining the wind direction and wi is a scalar modeling the wind intensity varying between 0.0 and 1.0. Figure 7 shows the deformation in the shape of stalactites given by random wi values at each ring creation and also random $\vec{w}$ due to the presence of air circulation in the interior of the cave.

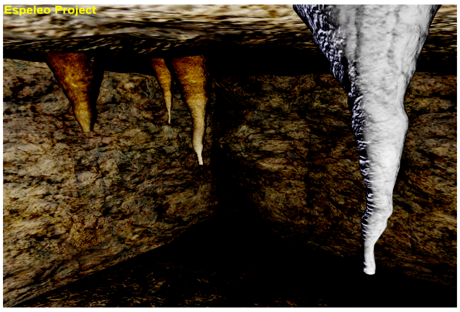

Figure 7: Effect of wind in the formation of stalactites.

The growth of speleothems is suspended due to four factors: *(i)* the pluviometric index is in Level 0; *(ii)* the *radius* of the new created ring is equal to zero;

(iiii) when the speleothem reaches the extremity of the cave, i.e. roof (stalagmites) or floor (stalactites) and *(iv)* when a *column* speleothem is created. When any of these conditions is satisfied, the vertical growth is suspended and only surface deformations and horizontal growth of the speleothems will occur.

This step completes the process of geometry amplification using the GS. However, in order for the simulated speleothem to reach an approximate shape of a real speleothem, it is necessary several successive passes in the pipeline so that its full shape is reached. At each pass, all the primitives created in the last pass are the new input for the geometry-shader stage. For that, the new primitive is sent back to the initial stages of the pipeline, using the stream-output stage. Before passing the result to the next pipeline stages (rasterizer and pixel-shader), the data is stored in a special buffer and sent to memory. The application takes charge of rendering what was created, and also, of sending back the same result to the initial stages of the pipeline. Figure 8 shows the evolution in the construction of the speleothems in the several successive passes through the pipeline before the final render.

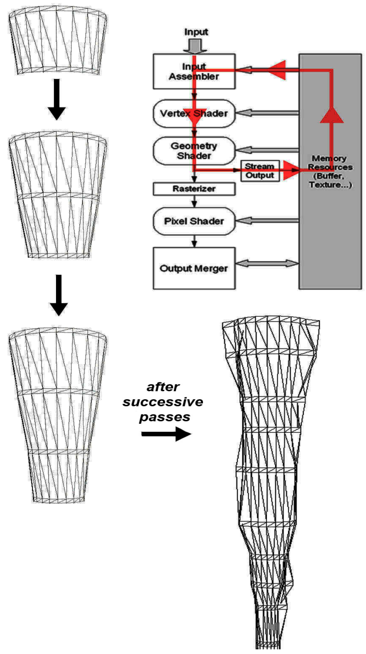

Figure 8: Growth of the speleothem. Successive passes through the pipeline

4.2 Appearance of Speleothems

Once we have the shape defined, we have to model the appearance of the speleothems. According to (Beynen et al., 2001), when visualized in visible or transmitted light, the speleothems show variations of hue and intensity. The standard coloration of the speleothems is white due to the limestone. However, other minerals and impurities present in limestone rock, can mix with the solution at the limestone dissolution stage. Depending on the type of mineral mixed, the speleothem will assume a characteristic color of the same. For instance, the presence of iron can produce ornamentations of red coloration, ocher, brown, orange, and yellow. Manganese oxides produce speleothems of black color, gray-bluish and deep and brilliant blue. Copper salts produce blue speleothems. The malachite produces blue-greenish coloration. In our model the user can choose the mineral type that will produce a specific coloration for the speleothem in formation. It is possible, on the fly, to change the rate of intensity of the mixture of the mineral in the solution, varying the color of speleothem. Since we do not have a model for color distribution along the body of the speleothem, we used a linear variation from base to tip. To calculate this variation in coloring (final vertex color), we use the equation: $fvc = mc * (arID/rc)$, where *arID* is the actual ring ID, *rc* is the amount of all the rings that the speleothem is made off and *mc* is the mineral color RGB data. The color will increase linearly according to the current ring we are building. For our implementation, since the literature on this subject does not present RGB data, we had to approximate the values from pictures of the real speleothems. The RGB data for each possible mineral is showed in Table 1.

Table 1: *Mineral Color (mc)*

Mineral Type	**RGB value**
Calcareous	(0.8, 0.8, 0.8)
Iron	(0.87, 0.62, 0.34)
Manganese	(0.32, 0.44, 0.54)
Copper	(0.05, 0.7, 0.7)
Malachite	(0.67, 0.82, 0.75)

Still in (Beynen et al., 2001), the crystals of calcite which form the body of the speleothems are translucent, thus providing great dispersion and reflection when exposed to any type of lighting, and fluorescence in the visible spectrum when exposed to UV light. Those optics properties are also attributed to the mixed impurities to the solution in the limestone dissolution stage. To improve the appearance of the speleothems and to simulate the properties mentioned above, we used standard graphics techniques such as Normal Mapping (Cohen et al., 1998; Cignoni et al., 1998), Tone Mapping (Ashikhmin, 2002) and Environment Mapping (Blinn and Newell, 1976; Miller and Hoffman, 1976). Normal Mapping is used to increase the effect of ruggedness on the surface of the speleothems, together with the horizontal displacement, described in Section 4.1. Tone Mapping and Environment Mapping are used for the effect of brightness and reflection, simulating the properties of crystals of calcita and moisture caused by aqueous solution that covers the speleothems.

We need successive passes for the shape of the speleothem to be completed. The actual number of passes needed is entirely dependent on the random values of the variables radius and height set by the application. For that reason, for each new execution of the application, we will have different speleothems. It is also possible to create the speleothems in a manual mode. The application interface allows to insert values for the parameters showed in the Table 2 through interactive controls. These controls can also be used to drive the creation of certain shapes of speleothems, instead of creating them in a random way. During the process of speleothems creation, is possible to do any manipulation in the application such as changing values of parameters, camera navigation and also set parameters values for the Normal Mapping, Tone Mapping and Environment Mapping techniques.

Table 2: *Parameters for speleothems growth*

PIL	**Drip Int.**	**CO2 Conc.**	**Type**	**Radius**	**Height**
0	-	-	-	-	-
1	-	low	stalactite	decrease x2	increase x1
1	-	normal	stalactite	decrease x1	increase x2
1	-	high	stalactite	same	increase x3
2	dry	low	stalagmite	decrease x1	increase x1
2	standard	low	stalagmite	same	increase x1
2	wet	low	stalagmite	increase x1	increase x1
2	dry	normal	stalagmite	decrease x1	increase x2
2	standard	normal	stalagmite	same	increase x2
2	wet	normal	stalagmite	increase x1	increase x2
2	dry	high	stalagmite	decrease x1	increase x3
2	standard	high	stalagmite	same	increase x3
2	wet	high	stalagmite	increase x1	increase x3

5 RESULTS

In this section we illustrate a few of our results. These were obtained on a Pentium D 2.8 GHz, 2GB RAM, NVIDIA GeForce 8600 GT. The application was implemented using the DirectX 10 API, HLSL (St-Laurent, 2005). Figure 9 summarizes our results. In (**a**) we show stalactites created by random values of radius and height in according to the parameters of Table 1. The values change at every application time step. In (**c**) we show a stalactite growth with LOW CO2 Concentration and all remaining parameters fixed. In (**b**) we show a stalactite growth with NORMAL CO2 Concentration and figure (**d**) shows a stalactite growth with HIGH CO2 Concentration in the drop of satured solutiuon. Figure 9 (**e**) shows stalagmites with Drip Interval (DRY) and NORMAL CO2 Concentration. The figure (**f**) shows columns created by joining stalagmites and stalactites. Figures (**g**) and (**h**) show both stalactites and stalagmites made with random values generated by the application on-the-fly. In the figures (**a, b, c, f**) the predominant mineral is calcareous, therefore the color of speleothems is white. In figures (**d, e, g**) the predominant mineral is iron. The color of speleothems vary between red and yellow. In figure (**h**) the predominant mineral is malachite which alters the color to blue-greenish. The frame rate of our implementation depends on the amount of speleothems being created. The amount is set in the beginning of the simulation. Compared with the work of (Yang and Ouhyoung, 1992), our model makes use of meaningful geological parameters to control the shape and appearance of the speleothems. Our approach improves the visual quality and also creates a simple shape with a low polygon count, taking advantage of the Geometry Shader. Our approach is plausible with the growth of real speleothems.

6 CONCLUSIONS

This work presented a simple model for growth of speleothems in real-time, using real geological parameters of growth of speleothems. We also used the powerful amplification capability of the geometry-shader and stream-output stages present in last generation GPUs. It is clear that the availability of these two graphical pipeline stages becomes a rich source for exploration of new forms to create geometry. However, there is a lot to explore on this subject, not only in the simulation of this specific natural phenomena that is rich in information, but also in the application side. We have not touched yet the full richness of the way speleothems reflect light, and this is left for future work. We assess that the final results were satisfactory compared with the real forms of speleothems, making it possible to generate a large variation of speleothems in realtime, suitable for graphics applications.

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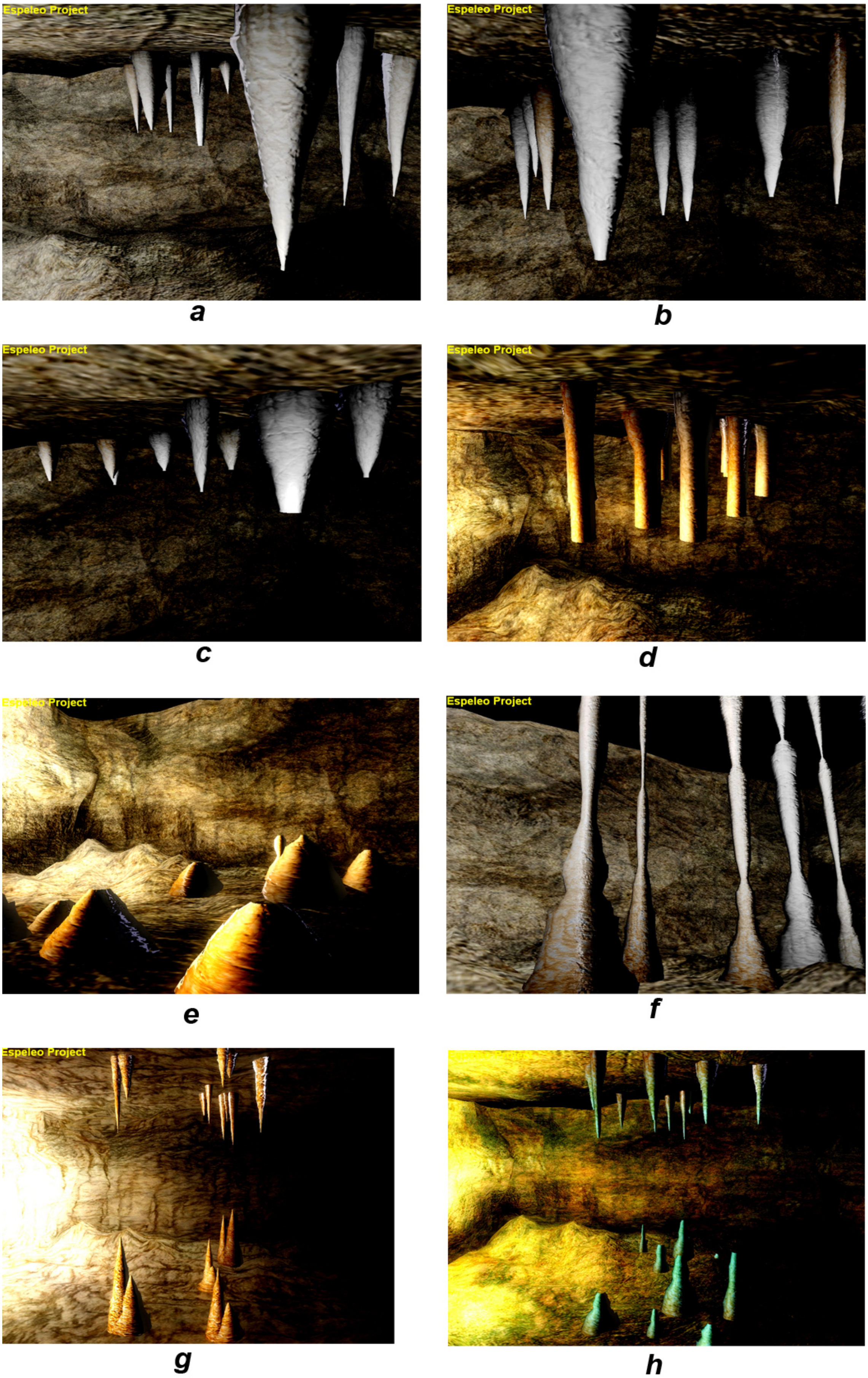

Figure 9: Virtual Speleothems.